Description Logics for Relative Terminologies

Szymon Klarman

Vrije Universiteit Amsterdam, Department of Computer Science,
De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
sklarman@few.vu.nl

Abstract. Context-sensitivity has been for long a subject of study in linguistics, logic and computer science. Recently the problem of reasoning with contextual knowledge has been picked up also by the Semantic Web community. In this paper we introduce a conservative extension to the Description Logic $\mathcal{ALC}$ which supports representation of ontologies containing *relative terms*, such as 'big' or 'tall', whose meaning depends on the choice of a particular *comparison class* (context). We define the language and investigate its computational properties, including the specification of a tableau-based decision procedure and complexity bounds.

1 Introduction

It is a commonplace observation that the same expressions might have different meanings when used in different contexts. A trivial example could be that of the concept The_Biggest. Figure 1 presents three snapshots of the same knowledge base which focus on different parts of the domain. The extension of the concept visibly varies across the three takes. Intuitively, there seem to be no contradiction in that individual Moscow is an instance of The_Biggest, when considered in the context of European cities, an instance of ¬The_Biggest,

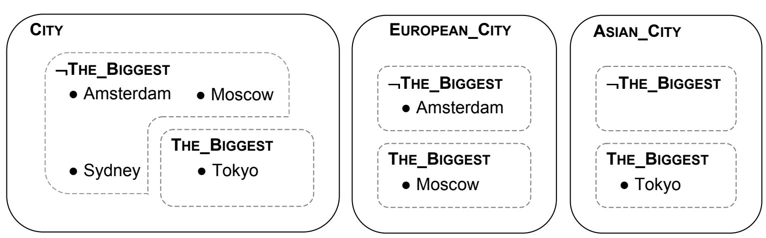

Fig. 1. Example of a relative concept The_Biggest.

when contrasted with all cities, and finally, an instance of none of these when the focus is only on the cities in Asia. Natural language users resolve such superficial incoherencies simply by recognizing that certain terms, call them *relative*, such as The_Biggest, acquire definite meanings only when put in the context of other denoting expressions[1] — in this case, expressions denoting so-called *comparison classes*, i.e. collections of objects with respect to which the terms are used [1,2].

The problem of context-sensitivity has been for long a subject of studies in linguistics, logic and even computer science. Recently, it has been also encountered in the research on the Semantic Web [3,4], where the need for representing and reasoning with imperfect information becomes ever more pressing. Relativity of meaning appears as one of common types of such imperfection. Alas, *Description Logics* (DLs), which form the foundation of the Web Ontology Language OWL [5], the basic knowledge representation formalism on the Semantic Web, were originally developed for modeling crisp, static and unambiguous knowledge, and as such, are incapable of handling the task seamlessly. Consequently, it has become highly desirable to look for more expressive, ideally backward compatible languages to meet the new application requirements [4,6].

In this paper we define a simple, conservative extension to the DL $\mathcal{ALC}$, which is intended for representing those relative terminologies, for which the nature of contextualization complies strictly to the following assumption:

$$\text{CONTEXT} = \text{COMPARISON CLASS}$$

This understanding of contexts, although very specific, is not uncommon in practical applications. In some domains, for instance geographical or medical, the use of qualitative descriptions involving relative terms is a typical way of escaping arbitrary threshold-based classification criteria. Technically, the adopted approach rests on a limited use of two-dimensional modal semantics [7], in which the basic object-oriented DL language can obtain multiple interpretations relative to possible worlds on a separate context dimension. Thus, scenarios like the one shown in Fig. 1 can be represented in an intuitive and elegant manner, conceptually and formally compatible with the model-theoretic paradigm of DLs.

This paper is a revised version of [8] and its follow-up [9]. The language presented here is considerably constrained with respect to the original proposal, while a much deeper study of its computational aspects is provided. In the next section we define the syntax and the semantics of the extension. Further, we specify a tableau-based decision procedure and derive complexity bounds on the satisfiability problem. In the last two sections, we shortly position our work in a broader perspective and conclude the presentation.

2 Representation Language

For a closer insight into the problem consider again the scenario from Fig. 1. Apparently, there is no straightforward way of modeling it in a standard DL fashion. Asserting both $\mathsf{Moscow} : \textsc{The\_Biggest}$ and $\mathsf{Moscow} : \neg\textsc{The\_Biggest}$ in the same knowledge base results in an immediate contradiction. On the other

[1] The philosophy of language qualifies them as *syncategorematic*, i.e. ones that do not form denoting expressions by themselves.

hand, using indices for marking versions of a concept in different contexts, such as in Moscow : THE_BIGGEST_EC and Moscow : ¬THE_BIGGEST_C, indeed allows to avoid inconsistency, but for the price of a full syntactic and semantic detachment of the indexed versions. Thus, the latter strategy makes it impossible to impose global constraints on the contextualized concepts, for instance, to declare that regardless of the context, THE_BIGGEST is always a subclass of BIG. Moreover, neither of the approaches facilitates use of knowledge about comparison classes per se, for instance, in order to infer contradiction in case EUROPEAN_CITY happens to be equivalent to CITY, and thus denote exactly the same context.

Finding a suitable fix for this kind of flaws motivates to a big extent our proposal. The logic $\mathcal{C}_{\mathcal{ALC}}$, introduced in this paper, extends the basic DL $\mathcal{ALC}$ with a modal-like operator which internalizes the use of comparison classes in the language. The classes are denoted by means of arbitrary DL concepts. Semantically, the operator is grounded in an extra modal dimension incorporated into DL interpretations, whose possible states are subsets of the object domain. We start by recalling the basic nomenclature of DLs and then give a detailed account of the syntax and the semantics of $\mathcal{C}_{\mathcal{ALC}}$.

2.1 Description Logic $\mathcal{ALC}$

A DL language is specified by a signature $\Sigma = (N_I, N_C, N_R)$, where N_I is a set of *individual names*, N_C a set of *concept names*, and N_R a set of *role names*, and a set of operators enabling construction of complex formulas [10]. The DL $\mathcal{ALC}$ permits concept descriptions defined by means of concept names (atomic concepts), special symbols $\top, \bot$ and the following constructors:

$$C, D, r \rightarrow \ \neg C \ | \ C \sqcap D \ | \ C \sqcup D \ | \ \exists r.C \ | \ \forall r.C$$

where C, D are arbitrary concept descriptions and r is a role. A knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ in $\mathcal{ALC}$, consists of the terminological and the assertional component. The (general) TBox $\mathcal{T}$ contains concept inclusion axioms $C \sqsubseteq D$ (abbreviated to $C \equiv D$ whenever $C \sqsubseteq D$ and $D \sqsubseteq C$). The ABox $\mathcal{A}$ contains axioms of possibly two forms: concept assertions $C(a)$ and role assertions $r(a, b)$, where a, b are individual names.

The semantics is defined in terms of an *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty *domain* of individuals, and $\cdot^{\mathcal{I}}$ is an *interpretation function*, which maps every $a \in N_I$ to an element of $\Delta^{\mathcal{I}}$, every $C \in N_C$ to a subset of $\Delta^{\mathcal{I}}$ and every $r \in N_R$ to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The function is inductively extended over complex terms in a usual way, according to the semantics of the operators. An interpretation $\mathcal{I}$ *satisfies* an axiom in either of the cases below:

- $\mathcal{I} \models C \sqsubseteq D$ *iff* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$,
- $\mathcal{I} \models C(a)$ *iff* $a^{\mathcal{I}} \in C^{\mathcal{I}}$,
- $\mathcal{I} \models r(a, b)$ *iff* $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in r^{\mathcal{I}}$.

An interpretation is a *model* of a knowledge base *iff* it satisfies all its axioms.

2.2 Description Logic $\mathcal{C}_{\mathcal{ALC}}$

The logic $\mathcal{C}_{\mathcal{ALC}}$ adds to the syntax of $\mathcal{ALC}$ a new concept constructor, based on modal-like *context operator* $\langle\cdot\rangle$:

$$C, D \rightarrow \langle D\rangle C$$

A *contextualized concept description* consists of a *relative concept* C and a specified *comparison class* D, which co-determines the meaning of C. Intuitively, $\langle D\rangle C$ denotes all objects which are C as considered in the context of all objects which are D. For instance, $\langle$CITY$\rangle$THE_BIGGEST describes the individuals that are the biggest as considered in the context of (all and only) cities. Other than that $\mathcal{C}_{\mathcal{ALC}}$ does not differ from $\mathcal{ALC}$ on the syntactic level.

Some deeper changes are introduced to the semantics of the language, which is augmented with an extra modal dimension, whose possible states — comparison classes/contexts — are defined extensionally as subsets of the (global) domain of interpretation. In each context a relevant part of the vocabulary is freely reinterpreted. Definition 1 introduces the notion of *context structure* which is an interpretation of a $\mathcal{C}_{\mathcal{ALC}}$ language.

Definition 1. *A* context structure *for a $\mathcal{C}_{\mathcal{ALC}}$ language is a triple $\mathcal{C} = \langle \Delta, W, \{\mathcal{I}_w\}_{w\in W}\rangle$, where:*

- Δ *is a* global domain *of interpretation,*
- $W \subseteq \wp(\Delta)$ *is a set of* comparison classes*, with $\Delta \in W$ and $\emptyset \notin W$,*
- $\mathcal{I}_w = (\Delta^{\mathcal{I}_w}, \cdot^{\mathcal{I}_w})$ *is an* interpretation *of the language in the context w:*
  - $\Delta^{\mathcal{I}_w} = w$ *is a non-empty domain of individuals,*
  - $\cdot^{\mathcal{I}_w}$ *is an interpretation function defined as usual.*

Given a context structure $\mathcal{C} = \langle \Delta, W, \{\mathcal{I}_w\}_{w\in W}\rangle$ we can now properly define the semantics of contextualized concept descriptions:

$$(\langle D\rangle C)^{\mathcal{I}_w} = \{x \in \Delta^{\mathcal{I}_w} \mid x \in D^{\mathcal{I}_w} \wedge x \in C^{\mathcal{I}_{w|D}}\}$$

where $w \mid D$ is an operation returning $v \in W$ such that $v = D^{\mathcal{I}_w}$. The accessibility relation over W, which we leave implicit, visibly follows the $\supseteq$-ordering of the comparison classes, with $\Delta \in W$ being its least element. Put differently, the context operator might give access only to a world whose domain is a subset of the current one. We also do not introduce the dual 'box' operator, as not very interesting from the modeling perspective and, moreover, practically redundant, even as an abbreviation for the usual $\neg\langle D\rangle\neg C$. Observe that according to our semantics $\neg\langle D\rangle\neg C = \neg D \sqcup \langle D\rangle C$, hence a $\mathcal{C}_{\mathcal{ALC}}$ formula in Negation Normal Form does not in fact contain negations in front of $\langle\cdot\rangle$.

For a finer-grained treatment of context-sensitivity we pose a few additional, natural constraints on the local interpretations of the vocabulary. First, we note that in general not the whole language should always be interpreted in a context, but only its part which is deemed meaningful in it. In our case, this is especially apparent with respect to individual names, which are in principle rigid, but in certain contexts might be loosing their designations. This phenomenon is sanctioned by the following assumption:

(RI) for every $a \in N_I$ and $w, v \in W$, if $a^{\mathcal{I}_w}$ and $a^{\mathcal{I}_v}$ are defined then $a^{\mathcal{I}_w} = a^{\mathcal{I}_v}$.

Further, we distinguish between *local* and *global* concept names (N_C^l and N_C^g, respectively) and roles (N_R^l and N_R^g). While the local terms (relative to contexts) are to be interpreted freely, the interpretations of the global ones (context-independent) are constrained so as to behave backward-monotonically along the accessibility relation:

(GC) for every $C \in N_C^g$ and $w \in W$, $C^{\mathcal{I}_w} = C^{\mathcal{I}_\Delta} \cap \Delta^{\mathcal{I}_w}$,
(GR) for every $r \in N_R^g$ and $w \in W$, $r^{\mathcal{I}_w} = r^{\mathcal{I}_\Delta} \cap \Delta^{\mathcal{I}_w} \times \Delta^{\mathcal{I}_w}$.

Finally, we allow *local* and *global* TBoxes ($\mathcal{T}^l$, $\mathcal{T}^g$). The global axioms hold universally in all contexts, whereas the local ones apply only to the root of the context structure. The intuition here is that some terminological constraints are analytical and thus context-independent (global), whereas others seize to hold when the focus shifts to a specific comparison class (local). For decidability reasons the syntax of global axioms is restricted to the $\mathcal{ALC}$ fragment. ABox axioms are left local in the above sense, although it is straightforward to extend their validity to all contexts by means of global vocabulary. As expected, the notion of satisfaction in $\mathcal{C}_{\mathcal{ALC}}$ is relativized to the context structure and a particular context in it, i.e. $\mathcal{C}, w \models \vartheta$ *iff* ϑ is satisfied by $\mathcal{I}_w$. A context structure $\mathcal{C}$ is a *model* of a knowledge base *iff* the constraints (**RI**), (**GC**), (**GR**) are respected in $\mathcal{C}$, and all the axioms are satisfied with respect to the following contexts:

- $\mathcal{C}, \Delta \models C \sqsubseteq D$, *if* $C \sqsubseteq D \in \mathcal{T}^l$,
- $\mathcal{C}, w \models C \sqsubseteq D$ for every $w \in W$, *if* $C \sqsubseteq D \in \mathcal{T}^g$,
- $\mathcal{C}, \Delta \models C(a)$,
- $\mathcal{C}, \Delta \models r(a, b)$.

It follows that both syntactically and semantically $\mathcal{C}_{\mathcal{ALC}}$ is a conservative extension of $\mathcal{ALC}$, i.e. an $\mathcal{ALC}$ knowledge base is satisfiable *iff* it is a satisfiable $\mathcal{C}_{\mathcal{ALC}}$ knowledge base.

2.3 Representation of Relative Terminologies

As an example of a $\mathcal{C}_{\mathcal{ALC}}$ knowledge base we will formalize a toy ontology of cities and towns and their relative sizes. On a larger scale, similar conceptualizations are common, for instance, in modeling geographic information, where not seldom are notions defined by means of relative terms referring to comparison classes. Such strategy allows to avoid the use of arbitrary value intervals on some physical attributes, and replace them by their qualitative and more practical approximations [11].

$$
\begin{aligned}
\mathcal{T}^l &= \{ \; (1)\ \textsc{City} \equiv \textsc{European\_City} \sqcup \textsc{Asian\_City}, \\
&\qquad (2)\ \textsc{European\_City} \sqcap \textsc{Asian\_City} \sqsubseteq \bot, \\
&\qquad (3)\ \textsc{Town} \equiv \langle \textsc{City} \rangle \textsc{Small} \; \} \\
\mathcal{T}^g &= \{ \; (4)\ \textsc{The\_Biggest} \sqsubseteq \textsc{Big}, \\
&\qquad (5)\ \textsc{Big} \sqcap \textsc{Small} \sqsubseteq \bot \; \} \\
\mathcal{A} &= \{ \; (6)\ \langle \textsc{City} \rangle \textsc{The\_Biggest}(\mathsf{Tokyo}), \\
&\qquad (7)\ \langle \textsc{Asian\_City} \rangle \textsc{The\_Biggest}(\mathsf{Tokyo}) \; \}
\end{aligned}
$$

We assume that concepts TOWN, CITY, EUROPEAN_CITY and ASIAN_CITY are to be interpreted globally, whereas the remaining ones locally. The local TBox states that every city is either a European or an Asian city (1), that these two classes are disjoint (2), and that towns are the small cities (3). Further, we ensure that regardless of the context, the biggest objects are always big (4), and these in turn are never the same as small (5). Finally, we assert that Tokyo is the biggest as compared to cities (6) and as compared to Asian cities (7). Given this setup it can be shown, for instance, that the following entailments hold:

$$\mathcal{K} \models \langle \text{CITY} \sqcap \neg\text{EUROPEAN\_CITY}\rangle \text{BIG}(\mathsf{Tokyo})$$
$$\mathcal{K} \models \neg\text{TOWN}(\mathsf{Tokyo})$$

The validity of the first entailment rests on the fact that Asian cities are exactly those that are cities but not the European ones (1,2). Hence, the comparison class denoted by ASIAN_CITY is the same as that described by CITY $\sqcap$ $\neg$EUROPEAN_CITY. Consequently, since Tokyo is an instance of THE_BIGGEST in the former context (7), this has to be the case as well in the latter. Finally, being the biggest there it has to be naturally an instance of BIG (4). By similar reasoning we can also demonstrate the second claim. Observe that Tokyo is an instance of BIG in the context of all cities (4,6), and therefore of $\neg$SMALL in that context (5). But then it follows that it cannot be a town, or else it would have to be a small city (3).

3 Reasoning with Comparison Classes

To properly frame the discussion over the computational aspects of $\mathcal{C}_{\mathcal{ALC}}$, we should first carefully consider the relationship between the syntactic and the semantic view on the contexts involved in our logic. Syntactically, every $\mathcal{C}_{\mathcal{ALC}}$ formula ϑ induces a finite tree of *context labels* $\Lambda_\vartheta = \{\gamma, \delta, \ldots\}$, isomorphic to the structure of $\langle\cdot\rangle$-nestings in the formula. For instance, the inclusion $\langle A\rangle\top \sqsubseteq \langle B\rangle(\langle A\rangle\top \sqcap \langle B\rangle\top)$ gives rise to the tree in Fig. 2. The labels are represented as

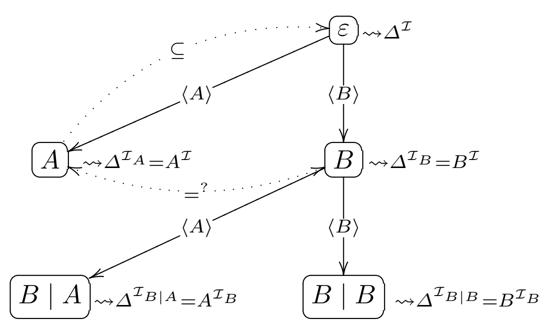

Fig. 2. A tree of context labels and the context structure.

finite sequences of concepts separated with vertical lines: $\gamma = C_1 \mid \ldots \mid C_{n-1} \mid C_n$, where every concept is the description occurring in some context operator on a certain depth of the formula. The empty label ε refers to the formula's root. Labels can be then easily rendered back into the language as $\mathcal{C}_{\mathcal{ALC}}$ concepts of the form $\gamma^{\mathcal{L}}\top$, where $\gamma^{\mathcal{L}} = \langle C_1\rangle \ldots \langle C_{n-1}\rangle\langle C_n\rangle$ for $\gamma = C_1 \mid \ldots \mid C_{n-1} \mid C_n$. Let us shift now to the semantic perspective. Clearly, the $\supseteq$-ordering of comparison classes in a context structure does not have to be necessarily tree-shaped. In fact, different descriptions of comparison classes might denote exactly the same subsets of the domain. This characteristic has to be appropriately handled in the reasoning. For that purpose we will find the following notion useful. An *assignment of context equalities* over a formula ϑ is a set $\Omega \subseteq \{\gamma \sim \delta \mid \gamma, \delta \in \Lambda_\vartheta\}$, where $\sim$ is an equivalence relation and Ω is closed under $\sim$.

3.1 Tableau decision procedure

The tableau calculus for $\mathcal{C}_{\mathcal{ALC}}$, presented in this section, is an extension of the well-known procedures for $\mathcal{ALC}$ [12]. The proof of satisfiability of a formula ϑ is a process of finding a complete and clash-free *constraint system* for ϑ (a set of logical constraints) by means of tableau rules. If such a system exists then ϑ is satisfiable — and unsatisfiable otherwise. The constraint systems are constructed by iterative application of inference rules to the constraints in the system.

Apart from variables for representing domain objects we also use context labels for marking contexts and assume that both sets are well-ordered by some relation $\ll$. By an abuse of notation we write $\gamma \in S$ (or $\gamma : x \in S$) to say that label γ (or term x within the scope of label γ) occurs in the system S. A proof for $\vartheta = \bigwedge_i \vartheta_i$, where every ϑ_i is a $\mathcal{C}_{\mathcal{ALC}}$ axiom, is initiated by setting a constraint system containing $\varepsilon : \vartheta_i$ for all i. For simplicity, we assume that every concept inclusion $\varepsilon : C \sqsubseteq D$ added to the tableau is instantaneously rewritten into an equivalent form $\varepsilon : \top \equiv \neg C \sqcup D$, similarly $\varepsilon : C \equiv D$ into $\varepsilon : \top \equiv (\neg C \sqcup D) \sqcap (C \sqcup \neg D)$ and all concept occurring on the tableau are in Negation Normal Form. Finally, we allow a special type of constraints $\gamma \sim \delta$, which represent designated context equalities.

The inference mechanism involves the standard $\mathcal{ALC}$ rules along with the $\mathcal{C}_{\mathcal{ALC}}$-specific rules, presented in the order of application in Tab. 1. The meaning of $\Rightarrow_{\langle\cdot\rangle}$ is straightforward: it introduces a relative concept assertion within the scope of a newly generated context label, thus marking a transition of the proof into a different context. The $\Rightarrow_\sim$ rule for every pair of different context labels occurring in the system decides non-deterministically whether the contexts denoted by them should be interpreted as equal or not. In either case respective constraints are added to the system to enforce generation of adequate models. Also, if the former is chosen, the rule introduces the corresponding equality statement over the context labels, which is used as a reference for application of the $\mathcal{ALC}$ rules. The rules $\Rightarrow_\subseteq$ and $\Rightarrow_\supseteq$ jointly ensure that for any context label $\gamma \mid C$ used in the system, a variable x occurs within its scope if and only if the system contains a constraint $\gamma : C(x)$. The remaining rules straightforwardly implement the semantics of global concepts, roles, and local and global TBox

$\Rightarrow_{\langle\cdot\rangle}$	**if** $\gamma : \langle C\rangle D(x) \in S$ **then set** $S' := S \cup \{\gamma \mid C : D(x)\}$
$\Rightarrow_{\sim}$	**if** $\{\gamma, \delta\} \subseteq S$, where $\gamma \neq \delta$, **then set** $S' := S \cup \{\varepsilon : (\gamma^{\mathcal{L}}\top \equiv \delta^{\mathcal{L}}\top)\} \cup \{\gamma \sim \delta\}$ **or** $S' := S \cup \{\varepsilon : (\gamma^{\mathcal{L}}\top \not\equiv \delta^{\mathcal{L}}\top)\}$
$\Rightarrow_{\subseteq}$	**if** $\gamma \mid C \in S$ **then set** $S' := S \cup \{\varepsilon : (\gamma^{\mathcal{L}}C \sqsubseteq \gamma^{\mathcal{L}}\langle C\rangle\top)\}$
$\Rightarrow_{\supseteq}$	**if** $\gamma \mid C : x \in S$ **then set** $S' := S \cup \{\gamma : C(x)\}$
$\Rightarrow_{C^g}$	**if** $\gamma : C(x) \in S$, where $C \in N_C^g$ **and** $\delta : x \in S$ **then set** $S' := S \cup \{\delta : C(x)\}$
$\Rightarrow_{R^g}$	**if** $\gamma : r(x, y) \in S$, where $r \in N_R^g$ **and** $\{\delta : x, \delta : y\} \subseteq S$ **then set** $S' := S \cup \{\delta : r(x, y)\}$
$\Rightarrow_{\equiv_{\mathcal{T}^l}}$	**if** $\varepsilon : (\top \equiv C) \in S$, where $\top \equiv C \in \mathcal{T}^l$ **and** $\varepsilon : x \in S$ **then set** $S' := S \cup \{\varepsilon : C(x)\}$
$\Rightarrow_{\equiv_{\mathcal{T}^g}}$	**if** $\varepsilon : (\top \equiv C) \in S$, where $\top \equiv C \in \mathcal{T}^g$ **and** $\gamma : x \in S$ **then set** $S' := S \cup \{\gamma : C(x)\}$
$\Rightarrow_{\not\equiv}$	**if** $\varepsilon : (C \not\equiv D) \in S$ **then** for a new $\ll$-minimal x **set** $S' := S \cup \{\varepsilon : C \sqcap \neg D(x)\}$ **or** $S' := S \cup \{\varepsilon : \neg C \sqcap D(x)\}$

Table 1. $\mathcal{C}_{\mathcal{ALC}}$ tableau rules.

axioms. The rule $\Rightarrow_{\not\equiv}$ applies only to the constraints introduced by $\Rightarrow_{\sim}$, which are interpreted locally.

The $\mathcal{ALC}$ rules ($\Rightarrow_{\sqcap}, \Rightarrow_{\sqcup}, \Rightarrow_{\exists}, \Rightarrow_{\forall}$, *blocking* and *clash* (*branch closure*), see [12]) are applied locally to the constraints with equal context labels, i.e. to the systems $S_\gamma = \{\phi(x) \mid \delta : \phi(x) \in S \text{ and } \delta \in [\gamma]\}$, where $[\gamma] = \{\delta \in S \mid \delta = \gamma \text{ or } \delta \sim \gamma \in S\}$. The constraints generated by a rule due to its application to the system S_γ are added to S with a $\ll$-minimal context label from $[\gamma]$. As usual it is required that application of the $\Rightarrow_{\exists}$ rule is deferred until no other rules apply. We say that S contains a clash if and only if there exists a label $\gamma \in S$ such that S_γ contains a clash. In such cases no other rules are applicable to S.

The correctness of the algorithm is proven in the appendix.

3.2 Computational complexity

It turns out that the convenient expressiveness of the language is compromised by a noticeable expense in the complexity of reasoning. More precisely, we are going

$(C \sqsubseteq D)_\varepsilon := C_\varepsilon \sqsubseteq D_\varepsilon$	*if* $C \sqsubseteq D \in \mathcal{T}^l$
$(C \sqsubseteq D)_\varepsilon := \bigcup_{\gamma \in \Lambda_\vartheta} (\gamma^{\mathcal{L}} C)_\varepsilon \sqsubseteq (\gamma^{\mathcal{L}} D)_\varepsilon$	*if* $C \sqsubseteq D \in \mathcal{T}^g$
$(C(a))_\varepsilon := C_\varepsilon(a)$	*if* $C(a) \in \mathcal{A}$
$(r(a,b))_\varepsilon := r_\varepsilon(a,b)$	*if* $r(a,b) \in \mathcal{A}$
$A_\gamma := A$ *if* $A \in N_C^g$	$(\neg A)_\gamma := \neg A_\gamma$
$A_\gamma := A^*$ *if* $A \in N_C^l$	$(C \sqcap D)_\gamma := C_\gamma \sqcap D_\gamma$
$A_\varepsilon := A$	$(C \sqcup D)_\gamma := C_\gamma \sqcup D_\gamma$
$\bot_\gamma := \bot$	$(\exists r.D)_\varepsilon := \exists r_\varepsilon.D_\varepsilon$
$\top_\gamma := \top$	$(\forall r.D)_\varepsilon := \forall r_\varepsilon.D_\varepsilon$
$r_\gamma := r$ *if* $r \in N_R^g$	$(\exists r.D)_{\gamma \mid C} := \exists r_{\gamma \mid C}.(\langle C \rangle_\gamma \sqcap D_{\gamma \mid C})$
$r_\gamma := r^*$ *if* $r \in N_R^l$	$(\forall r.D)_{\gamma \mid C} := \forall r_{\gamma \mid C}.(\langle C \rangle_\gamma \sqcap D_{\gamma \mid C})$
$r_\varepsilon := r$	$(\langle C \rangle D)_\gamma := \langle C \rangle_\gamma \sqcap D_{\gamma \mid C}$
$\langle C \rangle_{\gamma \mid D} := A^*$ and set: $\mathcal{T}' := \mathcal{T} \cup \{A^* \equiv \langle D \rangle_\gamma \sqcap C_{\gamma \mid D}\}$	
$\langle C \rangle_\varepsilon := A^*$ and set: $\mathcal{T}' := \mathcal{T} \cup \{A^* \equiv C_\varepsilon\}$	
for every $\langle C \rangle_\gamma$ and $\langle D \rangle_\delta$, with $\langle C \rangle_\gamma \neq \langle D \rangle_\delta$, set:	
$\mathcal{T}' := \mathcal{T} \cup \{\langle C \rangle_\gamma \equiv \langle D \rangle_\delta\}$ *iff* $(\gamma \mid C) \sim (\delta \mid D) \in \Omega$	
$\mathcal{T}' := \mathcal{T} \cup \{\langle C \rangle_\gamma \not\equiv \langle D \rangle_\delta\}$ *iff* $(\gamma \mid C) \sim (\delta \mid D) \notin \Omega$	

Table 2. Translation $\cdot_\varepsilon^\Omega$ from $\mathcal{C}_{\mathcal{ALC}}$ to $\mathcal{ALC}$ for a fixed Ω.

to show that any $\mathcal{C}_{\mathcal{ALC}}$ formula ϑ can be translated into an equisatisfiable $\mathcal{ALC}$ formula, which in the worst case is exponentially larger than ϑ. However, as the exponential blow-up stems exclusively from the fact that one has to account for all possible assignments of context equalities over the formula, we can therefore consider an 'oracle' providing a correct assignment and, as a result, obtain a translation only polynomially larger. Since the decision problem in $\mathcal{ALC}$ with non-empty TBoxes is ExpTime-complete [10], we will therefore conclude that the upper bound of deciding satisfiability in $\mathcal{C}_{\mathcal{ALC}}$ is NExpTime.

The translation of a formula $\vartheta = \bigwedge_i \vartheta_i$, where each ϑ_i is a $\mathcal{C}_{\mathcal{ALC}}$ axiom, is defined as:

$$\vartheta_\varepsilon = \bigvee_{\Omega \in \mathbf{\Omega}_\vartheta} \vartheta_\varepsilon^\Omega = \bigvee_{\Omega \in \mathbf{\Omega}_\vartheta} \bigwedge_i (\vartheta_i)_\varepsilon^\Omega$$

where $\mathbf{\Omega}_\vartheta$ is the set of all possible assignments of context equalities for ϑ. The details are presented in Tab. 2. The translation rests on introduction of fresh atoms, marked as A^* for new concept names and r^* for new role names, and supplementary TBox axioms, which constrain the interpretation of the added terms. Roughly, the new atoms are used to differentiate between occurrences of the same terms within non-equal contexts and additionally to abbreviate the references to the comparison classes. The following restrictions are imposed on the translation function $\cdot_\delta^\Omega$:

$$A_\gamma^\Omega = A_\delta^\Omega \text{ iff } \gamma \sim \delta \in \Omega$$
$$r_\gamma^\Omega = r_\delta^\Omega \text{ iff } \gamma \sim \delta \in \Omega$$

The following Lemma, which we prove in the appendix, states the general properties of the translation.

Lemma 1 (Translation properties). *For every $\mathcal{C}_{\mathcal{ALC}}$ formula ϑ it holds that:*

1. *ϑ is satisfiable* iff *ϑ_ε is satisfiable;*
2. *for a fixed assignment of context equalities Ω the size of $\vartheta_\varepsilon^\Omega$ is polynomial in the size of ϑ;*
3. *the size of ϑ_ε is exponential in the size of ϑ.*

Based on those we are able to derive the following complexity bounds for the satisfiability problem in $\mathcal{C}_{\mathcal{ALC}}$.

Theorem 1 (Complexity bounds). *The upper and the lower bound on the complexity of deciding satisfiability of a $\mathcal{C}_{\mathcal{ALC}}$ formula are* NExpTime *and* ExpTime*, respectively.*

As $\mathcal{C}_{\mathcal{ALC}}$ is a conservative extension of $\mathcal{ALC}$, the lower bound ExpTime carries over directly from $\mathcal{ALC}$ [10]. For the upper bound, note that Lemma 1 implies that under a fixed assignment of context equalities the size of the $\mathcal{ALC}$ formula resulting from the translation can be at most polynomially larger than that of the original $\mathcal{C}_{\mathcal{ALC}}$ formula. Assuming the correct assignment is given, solving the problem can be at most as hard as in $\mathcal{ALC}$ w.r.t. a non-empty TBox, i.e. ExpTime-complete. Deciding satisfiability in $\mathcal{C}_{\mathcal{ALC}}$ is therefore in NExpTime.

Apart from indicating the upper bound for complexity, Lemma 1 offers also two additional insights into $\mathcal{C}_{\mathcal{ALC}}$. Firstly, it provides an alternative to the tableau decision procedure. In fact, both approaches are very closely related. In particular, they involve the same exponential blow-up associated with the number of possible assignments of context equalities, and also the treatment of terms occurring within the scope of context operators is analogical in both cases. Regardless of that, it is worthwhile to study the tableau calculus independently, as some potential extensions of $\mathcal{C}_{\mathcal{ALC}}$ (e.g. allowing anonymous context operators) might easily impede the translation-based strategy, while still remain possible to handle by the tableau augmented with some additional rules.

Secondly, it shows that strictly speaking $\mathcal{C}_{\mathcal{ALC}}$ is not more expressive than $\mathcal{ALC}$. Nevertheless, there exists no equivalence-preserving reduction, i.e. formulas of $\mathcal{C}_{\mathcal{ALC}}$ do not have in general equivalent counterparts in $\mathcal{ALC}$. For this reason we conjecture that $\mathcal{C}_{\mathcal{ALC}}$ is strictly more succinct than $\mathcal{ALC}$, a feature very appealing for representation languages.

4 Related Work

The logic $\mathcal{C}_{\mathcal{ALC}}$ can be seen as a special case of *multi-dimensional DLs* [13], and more generally, as an instance of *multi-dimensional modal logics* [7], in which next to the standard object dimension we introduce a second one, referring to the subsets of the object domain as the possible states in the model. The scope of multi-dimensionality involved here, however, is very limited, thus discharging certain computational problems inherent to richer multi-dimensional formalisms. Notably, we were able to define a terminating decision procedure without resorting to some more advanced techniques such as based on *quasimodels* [14].

The problem of representing and reasoning with contextual knowledge, in particular in DLs, has been quite extensively studied in the literature, e.g. in [4,15,3,16,17]. The vast majority of authors, following the tradition of McCarthy and others [18,19,20], consider contexts on a very abstract level, as primitive First-Order objects, which by themselves do not have any properties. Thus the general semantic intuition of introducing an additional dimension, in an explicit (e.g. by listing all contexts [4,15,3]) or an implicit (e.g. by treating subsets of models of a knowledge base as contexts [16,17]) manner, is common with our approach. However, as the problem we address here is more specific, we are also able to offer a stronger explication of what a context is — namely a subset of domain objects — and, consequently a stronger inference mechanism. Some generality is therefore sacrificed for the sake of problem-specific customization.

Finally, $\mathcal{C}_{\mathcal{ALC}}$ shares certain similarities with *public announcement logic* (PAL) [21], which studies the dynamics of information flow in epistemic models. Interestingly, our context operator can be to some extent seen as a PAL announcement, whose role is to reduce the DL (epistemic) model to exactly those individuals (epistemic states) that satisfy given concept (formula). Unlike in PAL, however, we interpret an application of the operator as a leap to a different model, rather than an update of the current one, thus allowing for a change in the meaning of relative terms. Because of that, it is also not possible to reduce reasoning in $\mathcal{C}_{\mathcal{ALC}}$ to PAL, for which tableau proof procedures exist, e.g. [22], or directly transfer other interesting results [23]. Only in a special case (empty TBoxes and only global concepts and roles) is $\mathcal{C}_{\mathcal{ALC}}$ a variant of PAL on unrestricted frames.

5 Conclusions and Future Work

Providing a sound formal account of context-sensitivity and related phenomena is a vital challenge in the field of knowledge representation, and quite recently, also on the Semantic Web. In this paper we have addressed a very specific case of that problem, namely, representation of *relative terms*, whose meaning depends on the selection of comparison classes to which the terms are applied.

Admittedly, the scope of the proposal is quite narrow and it does not pretend to have solved the general problem of context-sensitivity in DL-based representations. Nevertheless, we have showed that by a careful use of supplementary modal dimensions one can obtain extra expressive power, which on the one hand is sufficient to handle certain interesting representation problems, while on the other does not require deep revisions on the syntactic, semantic nor, most importantly, the proof-theoretic side of the basic DL paradigm. Our belief, which we aim to verify in the course of the future work, is that in a similar manner, aspects of multi-dimensionality can offer convenient formal means for addressing other types of context-sensitivity, and other phenomena related to imperfect knowledge, such as *uncertainty* or *vagueness*, which currently are approached on the grounds of formalisms involving a thorough reconstruction of the semantics and the proof theory of DLs, e.g. probabilistic, possibilistic or fuzzy DLs [24].

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Appendix

We prove that the results of soundness, termination and completeness hold for the tableau calculus presented in Sec. 3. The notation and the structure of the proofs follow closely the presentation in [7]. In particular we use $con(\vartheta)$ to denote all subconcepts of ϑ, $rol(\vartheta)$ for role names, and $obj(\vartheta)$ for all individual names occurring in the formula.

Theorem 2 (Soundness). *Let S be a complete clash-free constraint system for ϑ. Then ϑ is satisfiable.*

Proof. We use S as 'a guide' to show that there exists a model for ϑ. First, we define a structure $\mathcal{C} = \langle \Delta, W, \{\mathcal{I}_w\}_{w\in W}\rangle$, using the following abbreviations $[\gamma] = \{\delta \in S \mid \delta = \gamma \text{ or } \delta \sim \gamma \in S\}$ and $S_\gamma = \{\phi(x) \mid \delta : \phi(x) \in S \text{ and } \delta \in [\gamma]\}$:

- $w_{[\gamma]} \in W$ for every $[\gamma]$, such that $\gamma \in S$, where $w_{[\gamma]} = \Delta^{\mathcal{I}_{[\gamma]}}$ comprises all terms x such that $x \in S_\gamma$
- $\Delta = \bigcup W$
- $x \in A^{\mathcal{I}_{[\gamma]}}$ for every γ, x and A such that $A(x) \in S_\gamma$
- $a^{\mathcal{I}_{[\gamma]}} = a$ for every γ and $a \in obj(\vartheta)$ such that $a \in S\gamma$
- $r^{\mathcal{I}_{[\gamma]}}(x, y)$ for every γ, x, y and r such that $r(x, y) \in S_\gamma$ (or $r(z, y) \in S_\gamma$ in case x is blocked in S_γ by z)

It follows that $\mathcal{C}$ is a context structure. Note, that we ensure: 1) $w_{[\gamma]} = w_{[\delta]}$ if and only if $[\gamma] = [\delta]$, by the closure of S under $\Rightarrow_{\sim}$, 2) the rigidity of individual names, by the construction of $\mathcal{C}$, 3) the semantics of global concept and role names, by the closure of S under $\Rightarrow_{C^g}$ and $\Rightarrow_{R^g}$. Next we prove the following:

Proposition 1. *For every $w_{[\gamma]} \in W$, concept $C \in con(\vartheta)$ and $x \in \Delta^{\mathcal{I}_{[\gamma]}}$, if $\gamma : C(x) \in S$ then $x \in C^{\mathcal{I}_{[\gamma]}}$.*

Proof. The proof is by induction on the form of C.

- For atomic concepts the claim follows directly from the definition of $\mathcal{C}$.
- Let $C = \neg A$ for some atomic concept A and suppose $\gamma : \neg A(x) \in S$ for any γ and x. Notice that there can be no $\delta : A(x) \in S$ for any $\delta \in [\gamma]$ as S is clash-free. Hence $x \notin A^{\mathcal{I}_{[\gamma]}}$ and $x \in (\neg A)^{\mathcal{I}_{[\gamma]}}$.
- The cases of $C = B \sqcap D$, $C = B \sqcup D$, $C = \exists r.D$ and $C = \forall r.D$ are as in $\mathcal{ALC}$ (see [7]), modulo proper indexing of the interpretation function and the labeling of contexts.
- Let $C = \langle B \rangle D$ and suppose $\gamma : \langle B \rangle D(x) \in S$. Then since S is closed under $\Rightarrow_{\langle\cdot\rangle}$, it has to be the case that $\gamma \mid B : D(x)$. But then there exists $w_{[\gamma|B]} \in W$, such that $x \in D^{\mathcal{I}_{[\gamma|B]}}$. Also, since S is closed under $\Rightarrow_{\subseteq}$ and $\Rightarrow_{\supseteq}$, it has to be the case that $\Delta^{\mathcal{I}_{[\gamma|B]}} = B^{\mathcal{I}_{[\gamma]}}$. Therefore $x \in (\langle B \rangle D)^{\mathcal{I}_{[\gamma]}}$. □

By Proposition 1 we can finally show the following:

Proposition 2. *$\mathcal{C}$ is a model of ϑ.*

Proof. Recall that $\vartheta = \bigwedge_i \vartheta_i$ and for all i, $\varepsilon : \vartheta_i$ is in the initial constraint system for ϑ with empty context labels. Observe that $w_{[\varepsilon]}$ is the root of $\mathcal{C}$ and consider possible syntactic form of any ϑ_i:

- $C(a)$: then by Prop. 1 $a \in C^{\mathcal{I}_{[\varepsilon]}}$, hence $\mathcal{C}, w_{[\varepsilon]} \models C(a)$
- $r(a,b)$: then by definition of $\mathcal{C}$ it holds that $\langle a^{\mathcal{I}_{[\varepsilon]}}, b^{\mathcal{I}_{[\varepsilon]}} \rangle \in r^{\mathcal{I}_{[\varepsilon]}}$ and hence $\mathcal{C}, w_{[\varepsilon]} \models r(a,b)$
- $\top \equiv C \in \mathcal{T}^l$: then since S is closed under $\Rightarrow_{\equiv_{\mathcal{T}^l}}$, it has to be the case that for all $x \in \Delta^{\mathcal{I}_{[\varepsilon]}}$, $\varepsilon : C(x) \in S$ and by Prop. 1, $x \in C^{\mathcal{I}_{[\varepsilon]}}$. Hence $\top^{\mathcal{I}_{[\varepsilon]}} = C^{\mathcal{I}_{[\varepsilon]}}$, and consequently $\mathcal{C}, w_{[\varepsilon]} \models \top \equiv C$.
- $\top \equiv C \in \mathcal{T}^g$: then since S is closed under $\Rightarrow_{\equiv_{\mathcal{T}^g}}$, it has to be the case that for all γ, $x \in \Delta^{\mathcal{I}_{[\gamma]}}$, $\gamma : C(x) \in S$ and by Prop. 1, $x \in C^{\mathcal{I}_{[\gamma]}}$. Hence $\top^{\mathcal{I}_{[\gamma]}} = C^{\mathcal{I}_{[\gamma]}}$, and consequently $\mathcal{C}, w_{[\gamma]} \models \top \equiv C$ for all $w_{[\gamma]} \in W$.

It follows that each conjunct of ϑ is satisfied by $\mathcal{C}$. □

This completes the proof of soundness. □

Theorem 3 (Termination). *There is no infinite sequence of inference steps via the tableau rules.*

Proof. Consider a formula ϑ in NNF. Clearly, there is only a finite number of $\langle\cdot\rangle$ operators used in ϑ, and hence, there can be only a finite number of unique context labels introduced in the tableau due to application of the $\Rightarrow_{\langle\cdot\rangle}$ rule. Given that, there can be also only finite number of inference steps via the rules $\Rightarrow_{\sim}$ and $\Rightarrow_{\subseteq}$, as well as via the $\Rightarrow_{\supseteq}$ rule for any individual variable. Note, that other than occurrences of $\langle\cdot\rangle$, ϑ does not contain any symbols from outside $\mathcal{ALC}$, hence the only problem for termination is posed by application of the $\Rightarrow_{\exists}$ rule (clearly, upon suspending it there can be always only a finite number of possible inference steps). But given a finite number of context labels it has to be the case that at some point the blocking rule applies, and all $\ll$-minimal individual variables occurring in S are blocked. Hence the tableau procedure for ϑ terminates in finite time. □

Theorem 4 (Completeness). *If ϑ is satisfiable then there exists a complete clash-free constraint system for ϑ.*

Proof. Let ϑ be a $\mathcal{C}_{\mathcal{ALC}}$ formula and $\mathcal{C} = \langle \Delta, W, \{\mathcal{I}_w\}_{w\in W}\rangle$ a context structure satisfying ϑ. We use $\mathcal{C}$ as an oracle in determining the construction of a complete clash-free constraint system for ϑ. We say that a constraint system S is compatible with $\mathcal{C}$ *iff* there exist mappings $\pi : \Lambda_\vartheta \mapsto W$ and $\sigma : \Lambda_I \mapsto \Delta$, where Λ_ϑ and Λ_I are the context labels and the individual terms occurring in S, such that the following conditions are satisfied:

- $\pi(\varepsilon) = \Delta$,
- $\mathcal{C}, \Delta \models \phi$, for every formula ϕ whenever $\varepsilon : \phi \in S$
- $\sigma(a) = a^{\mathcal{I}_w}$ for every $a \in obj(\vartheta)$ and $w \in W$;
- $\sigma(x) \in \Delta^{\mathcal{I}_{\pi(\gamma)}}$ whenever $\gamma : x \in S$;
- $\sigma(x) \in C^{\mathcal{I}_{\pi(\gamma)}}$ whenever $\gamma : C(x) \in S$;
- $\langle\sigma(x), \sigma(y)\rangle \in r^{\mathcal{I}_{\pi(\gamma)}}$ whenever $\gamma : r(x, y) \in S$.

Let S be a constraint system for ϑ compatible with $\mathcal{C}$. We show that if any of the tableau rules is applicable to S, then it can be applied in such a way that the resulting system S' is still compatible with $\mathcal{C}$.

- The cases of $\Rightarrow_{\sqcap}$, $\Rightarrow_{\sqcup}$, $\Rightarrow_{\exists}$, $\Rightarrow_{\forall}$, $\Rightarrow_{\equiv_{\mathcal{T}^g}}$, $\Rightarrow_{\equiv_{\mathcal{T}^l}}$, $\Rightarrow_{\not\equiv}$ are as in $\mathcal{ALC}$ (see [7]), modulo relativization of the rules to local constraint systems S_γ, for particular $\gamma \in \Lambda_\vartheta$. The mapping π remains unmodified. S' is compatible with $\mathcal{C}$.
- Suppose we apply $\Rightarrow_{\langle\cdot\rangle}$ to some $\gamma : \langle C\rangle D(x) \in S$. Then we obtain S' by adding $\gamma \mid C : D(x)$ to S. We set $\pi(\gamma \mid D) := w$ for $w \in W$ such that $w = C^{\mathcal{I}_{\pi(\gamma)}}$, and leave σ unmodified. S' is compatible with $\mathcal{C}$.
- Suppose we apply $\Rightarrow_{\sim}$ to some $\{\gamma, \delta\} \subseteq S$. It must be that either $\pi(\gamma) = \pi(\delta)$ or $\pi(\gamma) \neq \pi(\delta)$ in $\mathcal{C}$. We pick the correct one and obtain S' by adding $\varepsilon : \gamma^{\mathcal{L}}\top \equiv \delta^{\mathcal{L}}\top$ or $\varepsilon : \gamma^{\mathcal{L}}\top \not\equiv \delta^{\mathcal{L}}\top$ to S, respectively. Clearly the added formula has to be satisfied in $\mathcal{C}, \pi(\varepsilon)$. Both mappings remain unmodified. S' is compatible with $\mathcal{C}$.

– Suppose we apply $\Rightarrow_{\subseteq}$ to some $\gamma \mid C \in S$. It has to be the case that $C^{\mathcal{I}_{\pi(\gamma)}} = \Delta^{\mathcal{I}_{\pi(\gamma|C)}}$. We obtain S' by adding $\varepsilon : \gamma^{\mathcal{L}}C \equiv (\gamma \mid C)^{\mathcal{L}}\top$ to S, which has to be clearly satisfied in $\mathcal{C}, \pi(\varepsilon)$. Both mappings remain unmodified. S' is compatible with $\mathcal{C}$.
– Suppose we apply $\Rightarrow_{\supseteq}$ to some $\gamma : x \in S$. Then we obtain S' by adding $\varepsilon : \top(x)$ to S. Since $\sigma(x) \in \Delta^{\mathcal{I}_{\pi(\gamma)}}$ it must also hold that $\sigma(x) \in \Delta^{\mathcal{I}_{\pi(\varepsilon)}}$. Both mappings remain unmodified. S' is compatible with $\mathcal{C}$.
– Suppose we apply $\Rightarrow_{\equiv_{C^g}}$ (resp. $\Rightarrow_{\equiv_{R^g}}$) to some $\{\gamma : C(x), \delta : x\} \subseteq S$ (resp. $\{\gamma : r(x,y), \delta : x, \delta : y\} \subseteq S$). We obtain S' by adding $\delta : C(x)$ (resp. $\delta : r(x,y)$) to S. But since C is a global concept (r is a global role) it must be already that $\sigma(x) \in C^{\mathcal{I}_{\pi(\delta)}}$ (resp. $\langle \sigma(x), \sigma(y) \rangle \in r^{\mathcal{I}_{\pi(\delta)}}$. Both mappings remain unmodified. S' is compatible with $\mathcal{C}$.

By Theorem 3 the number of inferences applicable to S is finite, therefore at some point we obtain a complete constraint system, which is clearly clash-free. □

Proof of Lemma 1. Let ϑ be a $\mathcal{C}_{\mathcal{ALC}}$ formula and $\vartheta_\varepsilon = \bigvee_{\Omega \in \mathbf{\Omega}_\vartheta} \vartheta_\varepsilon^\Omega$ its translation to $\mathcal{ALC}$. Then:

Claim 1. ϑ is satisfiable *iff* ϑ_ε is satisfiable;

Proof. For proving this claim we establish a correspondence between the models of ϑ and ϑ_ε. For every set of context equivalences Ω over the context labels Λ_ϑ, we relate the context structures $\mathcal{C} = \langle \Delta, W, \{\mathcal{I}_w\}_{w \in W} \rangle$ of ϑ to the interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\vartheta_\varepsilon^\Omega$ according to the following constraints. For every $\gamma \in \Lambda_\vartheta$, concept name $A \in con(\vartheta_\varepsilon^\Omega)$, role name $r \in rol(\vartheta_\varepsilon^\Omega)$, individual name $a \in obj(\vartheta_\varepsilon^\Omega)$ and finally for every occurrence of a $\langle C \rangle$ operator in $\vartheta_\varepsilon^\Omega$ we set:

$$\begin{array}{rl|l} \top_\gamma^{\mathcal{I}} &= \top^{\mathcal{I}_{\pi(\varepsilon)}} & A_\gamma^{\mathcal{I}} = A^{\mathcal{I}_{\pi(\gamma)}} \text{ iff } A \in N_C^l \\ \bot_\gamma^{\mathcal{I}} &= \bot^{\mathcal{I}_{\pi(\varepsilon)}} & A_\gamma^{\mathcal{I}} = A^{\mathcal{I}_{\pi(\varepsilon)}} \text{ iff } A \in N_C^g \\ \Delta^{\mathcal{I}} &= \Delta & r_\gamma^{\mathcal{I}} = r^{\mathcal{I}_{\pi(\gamma)}} \text{ iff } r \in N_R^l \\ \langle C \rangle_\gamma^{\mathcal{I}} &= \Delta^{\mathcal{I}_{\pi(\gamma|C)}} & r_\gamma^{\mathcal{I}} = r^{\mathcal{I}_{\pi(\varepsilon)}} \text{ iff } r \in N_R^g \\ a^{\mathcal{I}} &= a^{\mathcal{I}_{\pi(\varepsilon)}} & \end{array}$$

where π is a mapping from Λ_ϑ to W such that $\pi(\varepsilon) = \Delta$ and for every $\gamma \in \Lambda_\vartheta$ it holds that $\pi(\gamma \mid C) = C^{\mathcal{I}_{\pi(\gamma)}}$. In what follows, to simplify the notation, we write $\cdot^{\mathcal{I}_\gamma}$ instead of $\cdot^{\mathcal{I}_{\pi(\gamma)}}$. Clearly, context structures uniquely determine the interpretations and vice versa.

The gist of the proof, which we demonstrate in the subsequent steps, lies in that for any $\mathcal{C}_{\mathcal{ALC}}$ concept C and its translation C_ε there exist corresponding interpretations $\mathcal{C} = \langle \Delta, W, \{\mathcal{I}_w\}_{w \in W} \rangle$ and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that $C^{\mathcal{I}_\varepsilon} = C_\varepsilon^{\mathcal{I}}$. Below, we call C a γ-concept whenever it translates to C_γ in ϑ_ε, for some $\gamma \in \Lambda_\vartheta$.

Proposition 3. *For every γ-concept P, such that P contains only concept names and symbols $\bot, \top, \neg, \sqcap, \sqcup$ it holds that $P^{\mathcal{I}_\gamma} = \Delta^{\mathcal{I}_\gamma} \cap P_\gamma^{\mathcal{I}}$.*

Proof. Transform P to conjunctive normal form $\bigwedge_i \bigvee_j L_{ij}$, so that every L_{ij} is an atom, negated atom, $\bot$ or $\top$. Then we have that $\Delta^{\mathcal{I}_\gamma} \cap (\bigwedge_i \bigvee_j L_{ij})^{\mathcal{I}}_\gamma = \Delta^{\mathcal{I}_\gamma} \cap \bigcap_i \bigcup_j {L_{ij}}^{\mathcal{I}}_\gamma = \bigcap_i \bigcup_j (\Delta^{\mathcal{I}_\gamma} \cap {L_{ij}}^{\mathcal{I}}_\gamma)$. For every possible form of L_{ij} we show that $\Delta^{\mathcal{I}_\gamma} \cap {L_{ij}}^{\mathcal{I}}_\gamma = {L_{ij}}^{\mathcal{I}_\gamma}$:

- $L_{ij} = A$: if A is local then by the correspondence $A^{\mathcal{I}}_\gamma = A^{\mathcal{I}_\gamma}$ and obviously $\Delta^{\mathcal{I}_\gamma} \cap A^{\mathcal{I}_\gamma} = A^{\mathcal{I}_\gamma}$. If A is global then by the correspondence $A^{\mathcal{I}}_\gamma = A^{\mathcal{I}_\varepsilon}$. By the semantics of global concepts $A^{\mathcal{I}_\gamma} = \Delta^{\mathcal{I}_\gamma} \cap A^{\mathcal{I}_\varepsilon}$.
- $L_{ij} = \neg A$: then $\Delta^{\mathcal{I}_\gamma} \cap (\neg A)^{\mathcal{I}}_\gamma = \Delta^{\mathcal{I}_\gamma} \cap (\Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}_\gamma) = (\Delta^{\mathcal{I}_\gamma} \cap \Delta^{\mathcal{I}}) \setminus (\Delta^{\mathcal{I}_\gamma} \cap A^{\mathcal{I}}_\gamma)$ Hence, by the fact that $\Delta^{\mathcal{I}_\gamma} \subseteq \Delta^{\mathcal{I}}$ and by the previous argument this is equivalent to $\Delta^{\mathcal{I}_\gamma} \setminus A^{\mathcal{I}_\gamma}$ and thus to $(\neg A)^{\mathcal{I}_\gamma}$.
- $L_{ij} = \bot$: then by the correspondence $\Delta^{\mathcal{I}_\gamma} \cap \bot^{\mathcal{I}}_\gamma = \Delta^{\mathcal{I}_\gamma} \cap \bot^{\mathcal{I}_\varepsilon} = \emptyset = \bot^{\mathcal{I}_\gamma}$.
- $L_{ij} = \top$: then by the correspondence $\Delta^{\mathcal{I}_\gamma} \cap \top^{\mathcal{I}}_\gamma = \Delta^{\mathcal{I}_\gamma} \cap \top^{\mathcal{I}_\varepsilon} = \Delta^{\mathcal{I}_\gamma} = \top^{\mathcal{I}_\gamma}$.

Hence $\bigcap_i \bigcup_j (\Delta^{\mathcal{I}_\gamma} \cap {L_{ij}}^{\mathcal{I}}_\gamma) = \bigcap_i \bigcup_j {L_{ij}}^{\mathcal{I}_\gamma} = (\bigwedge_i \bigvee_j L_{ij})^{\mathcal{I}_\gamma} = P^{\mathcal{I}_\gamma}$, which concludes the proof. □

Proposition 4. *For every γ-concept $\exists r.P$ and $\forall r.P$, where P is as defined in Proposition 3, it holds that $\Delta^{\mathcal{I}_\gamma} \cap (\exists r.P)^{\mathcal{I}}_\gamma = (\exists r.P)^{\mathcal{I}_\gamma}$ and $\Delta^{\mathcal{I}_\gamma} \cap (\forall r.P)^{\mathcal{I}}_\gamma = (\forall r.P)^{\mathcal{I}_\gamma}$.*

Proof. By the translation and the correspondence we have that $\Delta^{\mathcal{I}_\gamma} \cap (\exists r.P)^{\mathcal{I}}_\gamma = \Delta^{\mathcal{I}_\gamma} \cap \{x \mid \exists y : r^{\mathcal{I}}_\gamma(x,y) \wedge y \in (\Delta^{\mathcal{I}_\gamma} \cap P^{\mathcal{I}}_\gamma)\}$ and analogically for $\forall r.P$. By Proposition 3 $\Delta^{\mathcal{I}_\gamma} \cap P^{\mathcal{I}}_\gamma = P^{\mathcal{I}_\gamma}$. Further, by the correspondence, if r is local then $r^{\mathcal{I}}_\gamma = r^{\mathcal{I}_\gamma}$ which is equivalent to $r^{\mathcal{I}_\gamma} \cap \Delta^{\mathcal{I}_\gamma} \times \Delta^{\mathcal{I}_\gamma}$, else if r is global we have $r^{\mathcal{I}}_\gamma = r^{\mathcal{I}_\varepsilon}$, but then $r^{\mathcal{I}_\varepsilon} \cap \Delta^{\mathcal{I}_\gamma} \times \Delta^{\mathcal{I}_\gamma} = r^{\mathcal{I}_\gamma}$. Hence $\Delta^{\mathcal{I}_\gamma} \cap \{x \mid \exists y : r^{\mathcal{I}}_\gamma(x,y) \wedge y \in (\Delta^{\mathcal{I}_\gamma} \cap P^{\mathcal{I}}_\gamma)\} = \{x \mid \exists y : r^{\mathcal{I}_\gamma}(x,y) \wedge y \in P^{\mathcal{I}_\gamma}\} = (\exists r.P)^{\mathcal{I}_\gamma}$ and analogically for $\forall r.P$. □

Proposition 5. *For every $\mathcal{ALC}$ γ-concept C it holds that $\Delta^{\mathcal{I}_\gamma} \cap C^{\mathcal{I}}_\gamma = C^{\mathcal{I}_\gamma}$*

Proof. Given the translation rules and the correspondence, the claim follows inductively from Propositions 3, 4 and the principle of compositionality. □

Proposition 6. *For every γ-concept $\langle C \rangle D$ it holds that $(\langle C \rangle D)^{\mathcal{I}}_\gamma = (\langle C \rangle D)^{\mathcal{I}_\gamma}$.*

Proof. By induction over the structure of D. Let D be an $\mathcal{ALC}$ concept. Then by the translation we get that $(\langle C \rangle D)^{\mathcal{I}}_\gamma = \langle C \rangle^{\mathcal{I}}_\gamma \sqcap D^{\mathcal{I}}_{\gamma|C}$, which by the correspondence is equivalent to $\Delta^{\mathcal{I}_{\gamma|C}} \cap D^{\mathcal{I}}_{\gamma|C}$ and by Proposition 5 to $D^{\mathcal{I}_{\gamma|C}} = (\langle C \rangle D)^{\mathcal{I}_\gamma}$. Assume D is any $\mathcal{C}_{\mathcal{ALC}}$ concept. Then given the translation rules and the correspondence, the claim follows inductively from Proposition 5, the former argument, and the principle of compositionality. □

Lemma 2. *For every ε-concept C it holds that $C^{\mathcal{I}}_\varepsilon = C^{\mathcal{I}_\varepsilon}$.*

Proof. Given the translation rules and the correspondence, the claim follows inductively from Propositions 5, 6, and the principle of compositionality. □

By comparing the semantics of $\mathcal{C}_{\mathcal{ALC}}$ and $\mathcal{ALC}$ axioms it follows immediately that if there exists a context structure satisfying a formula ϑ, then there has to exist an assignment Ω and an $\mathcal{ALC}$ model of a formula $\vartheta_\varepsilon^\Omega$, namely the one based on the correspondence:

- for TBox axioms and concept assertions directly by Lemma 2;
- for role assertions by the correspondence.

Likewise, the conditional holds in the opposite direction. Therefore, ϑ is satisfiable if and only if ϑ_ε is. □

Claim 2. for a fixed assignment of context equalities Ω the size of $\vartheta_\varepsilon^\Omega$ is polynomial in the size of ϑ;

Proof. The formula ϑ induces a finite number of context labels Λ_ϑ which is bounded by $|\vartheta|$. According to the translation rules for a fixed assignment of context equivalences Ω we introduce at most $|\Lambda_\vartheta|$ new concept names for representing the labels. Further the input is extended with:

- $|\Lambda_\vartheta|$ new TBox axioms, of length bounded by $|\vartheta|$, defining the new concept names;
- $\frac{|\Lambda_\vartheta|^2-|\Lambda_\vartheta|}{2}$ new TBox axioms, of constant length, for representing context equivalences;
- $|\Lambda_\vartheta|$ new TBox axioms of length bounded by $|\vartheta|$ for every global TBox axiom in ϑ;
- a number of new occurrences of concept names representing the contexts, linear in $|\vartheta|$.

It follows that the increase in the size of the translation $\vartheta_\varepsilon^\Omega$ is polynomial in the size of the original formula. □

Claim 3. the size of ϑ_ε is exponential in the size of ϑ.

Proof. By Point 2 there has to exist a polynomial function p such that $|\vartheta_\varepsilon^\Omega| = p(|\vartheta|)$. However, The whole translation ϑ_ε consists of $|\mathbf{\Omega}_\vartheta|$ different disjuncts of length $|\vartheta_\varepsilon^\Omega|$, where $\mathbf{\Omega}_\vartheta$ is the set of all possible assignments of context equivalences over Λ_ϑ. One can see that the size of $\mathbf{\Omega}_\vartheta$ is equal to $B(|\Lambda_\vartheta|)$, where B is a function computing so-called *Bell numbers* (the number of possible partitions of a set of a given cardinality). It can be shown that for any $|\Lambda_\vartheta| > 1$ it holds that $2^{|\Lambda_\vartheta|} \leq B(|\Lambda_\vartheta|) < 2^{|\Lambda_\vartheta|^2}$. Therefore, since $|\Lambda_\vartheta|$ is bounded by $|\vartheta|$ there exists a polynomial function q, of the degree d, such that $|\vartheta_\varepsilon| = 2^{q(|\vartheta|)}$ and hence $|\vartheta_\varepsilon| \in O(2^{(|\vartheta|^d)})$. □