Automated Reasoning in Artificial Intelligence:

**INTRODUCTION TO DESCRIPTION LOGIC**

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*(part of the content based on the tutorial by Stefan Schlobach)*

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Plan for today

- Description Logic knowledge bases
- Representing knowledge bases in Protégé
- Reasoning tasks and their reduction
**$\mathcal{ALC}$: syntax and semantics**

**Syntax:**
- **concept names**: $A, B, C \ldots$; e.g.: Man, Parent, Car,
- **role names**: $r, s \ldots$; e.g.: biggerThan, likes, locatedIn,
- **concept constructors**: $\top, A, \neg C, C \cap D, C \cup D, \exists r. C, \forall r. C$,
- **individual names**: $a, b \ldots$; e.g.: john, europe, snoopy.

**Semantics:**
An interpretation is a pair $\mathcal{I} = (\Delta \mathcal{I}, \cdot \mathcal{I})$, where $\Delta \mathcal{I}$ is a non-empty domain of individuals and $\cdot \mathcal{I}$ is an interpretation function, which maps:

- $\top \mathcal{I} = \Delta \mathcal{I}$,
- $A \mathcal{I} \subseteq \Delta \mathcal{I}$ for every concept name $A$,
- $r \mathcal{I} \subseteq \Delta \mathcal{I} \times \Delta \mathcal{I}$ for every role name $r$,
- $\cdot \mathcal{I}$ is extended inductively over complex concepts,
- $a \mathcal{I} \in \Delta \mathcal{I}$ for every individual name $a$. 
A DL knowledge base (alt. ontology) $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ consists of:

- **TBox** $\mathcal{T}$, i.e. terminology,
- **ABox** $\mathcal{A}$, i.e. assertions about individuals.
TBox: Syntax

Knowledge about relationships between concepts is expressed by means of *terminological axioms* (TBox axioms):

- **concept inclusion**: $C \sqsubseteq D$
  - necessary conditions for objects of type $C$.
  - Examples:
    
    \[\text{Elephant} \sqsubseteq \text{Animal} \cap \neg \text{Mouse}\]
    
    \[\text{Rich} \cap \text{Famous} \sqsubseteq \exists \text{knows}.(\text{Rich} \cap \text{Famous})\]

- **concept equivalence**: $C \equiv D$ (short for $C \sqsubseteq D$ and $D \sqsubseteq C$)
  - necessary and sufficient conditions for objects of type $C$
  - Examples:
    
    \[\text{Animal} \cap \text{Rational} \equiv \text{Man} \sqcup \text{Woman}\]
    
    \[\text{Person} \equiv \exists \text{hasParent}.\text{Person}\]

The *TBox* $\mathcal{T}$ of a KB is a finite set of terminological axioms.
Exercise: modeling TBoxes

An artist is someone who created an artwork. A sculpture is an artwork. A painting is an artwork that is not a sculpture. A painter is someone who created a painting. A sculptor is someone who created an artwork and created only sculptures. If an artwork is created by an artist, he has either painted or sculptured it. A multi-talent is both a painter and sculptor.

Model the information as a DL TBox:

Solution:
**Exercise: modeling TBoxes**

An artist is someone who created an artwork. A sculpture is an artwork. A painting is an artwork that is not a sculpture. A painter is someone who created a painting. A sculptor is someone who created an artwork and created only sculptures. If an artwork is created by an artist, he has either painted or sculptured it. A multi-talent is both a painter and sculptor.

Model the information as a DL TBox:

Solution:

\[
\begin{align*}
\text{Artist} & \equiv \exists \text{created}.\text{Artwork} \\
\text{Sculpture} & \sqsubseteq \text{Artwork} \\
\text{Painting} & \equiv \text{Artwork} \sqcap \neg \text{Sculpture} \\
\text{Painter} & \equiv \exists \text{created}.\text{Painting} \\
\text{Sculptor} & \equiv \exists \text{created}.\text{Artwork} \sqcap \forall \text{created}.\text{Sculpture} \\
\text{Artwork} & \sqsubseteq \exists \text{painted by}.\text{Artist} \sqcup \exists \text{sculptured by}.\text{Artist} \\
\text{Multitalent} & \sqsubseteq \text{Painter} \sqcap \text{Sculptor}
\end{align*}
\]
**ABox: Syntax**

Knowledge about individuals in the domain expressed in terms of the vocabulary is specified by means of *assertional axioms* (ABox axioms):

- **concept assertion:** $a : C$
  - individual $a$ is an instance of concept $C$
  - Example:
    
    - $mary : Mother$
    - $john : Rich \sqcup \exists \text{hasParent}.Rich$

- **role assertions:** $(a, b) : r$
  - individual $a$ is related to $b$ through the role $r$
  - Example:
    
    - $(john, mary) : \text{likes}$
    - $(new\_york, amsterdam) : \text{biggerThan}$

The *ABox* $\mathcal{A}$ of a KB is a finite set of assertional axioms.
Exercise: modeling ABoxes

Rembrandt created the artwork: “nightwatch”, but never created a sculpture. “nightwatch” is a painting. Michelangelo created at least one sculpture.

Model the information as a DL ABox.

Solution:
Exercise: modeling ABoxes

Rembrandt created the artwork: “nightwatch”, but never created a sculpture. “nightwatch” is a painting. Michelangelo created at least one sculpture.

Model the information as a DL ABox.

Solution:

\[(\text{rembrandt}, \text{nightwatch}) : \text{created} \]
\[
\text{rembrandt} : -\exists \text{created.} \text{Sculpture} \\
\text{nightwatch} : \text{Painting} \\
\text{michelangelo} : \exists \text{created.} \text{Sculpture}
\]
Protégé

Protégé is an ontology editor for OWL. But since OWL is a syntactic variant of DLs, OWL ontologies can be seen as DL knowledge bases.

DL vs. Protégé interface:

- OWL nomenclature: concept $\rightsquigarrow$ class, role $\rightsquigarrow$ object property.
- Protégé user-friendly syntax:

\[
\begin{array}{l}
\top \rightsquigarrow \text{Thing} \\
\bot \rightsquigarrow \text{Nothing} \\
\neg C \rightsquigarrow \text{not } C
\end{array}
\]

\[
\begin{array}{l}
C \sqsubseteq D \rightsquigarrow (C \text{ and } D) \\
C \sqcup D \rightsquigarrow (C \text{ or } D) \\
\exists r.C \rightsquigarrow (r \text{ some } C) \\
\forall r.C \rightsquigarrow (r \text{ only } C)
\end{array}
\]

- Designated fields in a template for entering axioms:

\[
\begin{array}{l}
C \sqsubseteq D \rightsquigarrow \text{Classes} / “C” / \text{Superclasses} / “D” \\
C \equiv D \rightsquigarrow \text{Classes} / “C” / \text{Equivalent classes} / “D” \\
a : C \rightsquigarrow \text{Individuals} / “a” / \text{Types} / “C” \\
(a, b) : r \rightsquigarrow \text{Individuals} / “a” / \text{Object prop. asser.} / “r” / “b”
\end{array}
\]
Exercise: modeling ontology in Protégé:

Enter the following KB into Protégé:

\[
\begin{align*}
\text{Artist} & \equiv \exists \text{created. Artwork} \\
\text{Sculpture} & \sqsubseteq \text{Artwork} \\
\text{Painting} & \equiv \text{Artwork} \sqcap \neg \text{Sculpture} \\
\text{Painter} & \equiv \exists \text{created. Painting} \\
\text{Sculptor} & \equiv \exists \text{created. T} \sqcap \forall \text{created. Sculpture} \\
\text{Multitalent} & \sqsubseteq \text{Painter} \sqcap \text{Sculptor}
\end{align*}
\]

\[\begin{align*}
\text{(rembrandt, nightwatch)} : \text{created} \\
\text{rembrandt} : \neg \exists \text{created. Sculpture} \\
\text{nightwatch : Artwork} \\
\text{michelangelo : } \exists \text{created. Sculpture}
\end{align*}\]
TBox: Semantics

Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be an interpretation. $\mathcal{I}$ satisfies a terminological axiom in either of the two cases:

- for $C \sqsubseteq D$ if and only $C^\mathcal{I} \subseteq D^\mathcal{I}$
- for $C \equiv D$ if and only $C^\mathcal{I} = D^\mathcal{I}$

An interpretation $\mathcal{I}$ is a *model* of the TBox $\mathcal{T}$ iff it satisfies every terminological axiom in $\mathcal{T}$. 
**TBox: Semantics example**

Let $\mathcal{I}$ be defined as:

- $\Delta^\mathcal{I} = \{\text{rembrandt, michelangelo, rodin, nightwatch, david, sixtChappel, thinker}\}$

- $\text{Artwork}^\mathcal{I} = \{\text{nightwatch, sixtChappel, thinker, david}\}$,
  $\text{Sculptor}^\mathcal{I} = \{\text{rodin, michelangelo}\}$
  $\text{Sculpture}^\mathcal{I} = \{\text{thinker, david}\}$
  $\text{Painter}^\mathcal{I} = \{\text{rembrandt, michelangelo}\}$
  $\text{Painting}^\mathcal{I} = \{\text{nightwatch, sixtChappel}\}$
  $\text{sculptured}^\mathcal{I} = \{(\text{rodin, thinker}), (\text{michelangelo, david})\}$
  $\text{created}^\mathcal{I} = \{(\text{rembrandt, nightwatch}), (\text{michelangelo, sixtChappel}), (\text{michelangelo, david}), (\text{rodin, thinker})\}$

Is $\mathcal{I}$ a model of $\mathcal{T}$?

- $\text{Painting} \sqsubseteq \text{Artwork} \sqcap \neg \text{Sculpture}$
- $\text{Painter} \equiv \exists \text{created}. \text{Painting}$
- $\text{Sculptor} \equiv \exists \text{sculptured}. \text{Artwork} \sqcap \forall \text{created}. \text{Sculpture}$
Reasoning tasks for TBoxes

For a TBox $\mathcal{T}$ and concepts $C, D$ occurring in $\mathcal{T}$.

**concept satisfiability:**

$C$ is *satisfiable* w.r.t. $\mathcal{T}$ iff there is a model $\mathcal{I}$ of $\mathcal{T}$: $C^\mathcal{I} \neq \emptyset$

**subsumption:** $\mathcal{T} \models C \sqsubseteq D$?

$C$ is *subsumed* by $D$ in $\mathcal{T}$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$ in every model $\mathcal{I}$ of $\mathcal{T}$

**equivalence:** $\mathcal{T} \models C \equiv D$?

Concepts $C$ and $D$ are *equivalent* in $\mathcal{T}$ iff $C^\mathcal{I} = D^\mathcal{I}$ in every model $\mathcal{I}$ of $\mathcal{T}$. 

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Reduction of TBox reasoning tasks

All TBox problems in $\mathcal{ALC}$ are reducible to concept satisfiability:

- $C$ is subsumed by $D$ in $\mathcal{T} \iff C \sqcap \neg D$ is unsatisfiable w.r.t. $\mathcal{T}$

  Proof:  \[ C \text{ is subsumed by } D \text{ in } \mathcal{T} \iff \text{for every model } \mathcal{I} \text{ of } \mathcal{T} \text{ it holds that } C^\mathcal{I} \subseteq D^\mathcal{I} \iff \text{for every model } \mathcal{I} \text{ of } \mathcal{T} \text{ it holds that } C^\mathcal{I} \cap (\neg D)^\mathcal{I} = \emptyset \iff \text{there is no model } \mathcal{I} \text{ of } \mathcal{T} \text{ s.t. } (C \sqcap \neg D)^\mathcal{I} \neq \emptyset \iff C \sqcap \neg D \text{ is unsatisfiable w.r.t. } \mathcal{T}. \]

- $C$ and $D$ are equivalent in $\mathcal{T} \iff C$ is subsumed by $D$ in $\mathcal{T}$ and $D$ is subsumed by $C$ in $\mathcal{T}$

Note: Notice that reduction to concept satisfiability requires: $\sqcap$ and $\neg$ for complex concepts.
ABox: Semantics

Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be an interpretation. $\mathcal{I}$ satisfies an assertional axiom in either of the two cases:

- for $a : C$ if and only $a^\mathcal{I} \in C^\mathcal{I}$
- for $(a, b) : r$ if and only $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$

An interpretation $\mathcal{I}$ is a model of the ABox $\mathcal{A}$ iff it satisfies every assertional axiom in $\mathcal{A}$.

An interpretation $\mathcal{I}$ is a model of $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ iff $\mathcal{I}$ is a model of both $\mathcal{A}$ and $\mathcal{T}$.

Note: By default Unique Name Assumption applies in DLs (but not in OWL!)
ABox: Semantics example

Let $\mathcal{I}$ be defined as:

- $\Delta^\mathcal{I} = \{\text{rembrandt, michelangelo, rodin, nightwatch, david, sixtChappel, thinker}\}$
- $\text{Artwork}^\mathcal{I} = \{\text{nightwatch, sixtChappel, thinker, david}\}$,
  $\text{Sculpture}^\mathcal{I} = \{\text{thinker, david}\}$
  $\text{Painting}^\mathcal{I} = \{\text{nightwatch, sixtChappel}\}$
  $\text{created}^\mathcal{I} = \{(\text{rembrandt, nightwatch}), (\text{michelangelo, sixtChappel}), (\text{michelangelo, david}), (\text{rodin, thinker})\}$

Is $\mathcal{I}$ a model of $\mathcal{A}$?

$(\text{rodin, thinker}) : \text{created}$

$\text{nightwatch} : \text{Artwork}$

$\text{rembrandt} : \neg \exists \text{created}. \text{Sculpture}$
Reasoning tasks for ABoxes

For a KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, a concept $C$, a role $r$ and individuals $a, b$:

**ABox consistency:**

$\mathcal{A}$ is *consistent* w.r.t $\mathcal{T}$ iff there is a model of $\mathcal{K}$.

*Note:* in such case we also say that $\mathcal{K}$ is *satisfiable*.

**Instance checking:** $\mathcal{K} \models a : C$ ? (resp. $\mathcal{K} \models (a, b) : r$) ?

- $a$ is *an instance* of $C$ in $\mathcal{K}$ iff every model of $\mathcal{K}$ is a model of $a : C$
- $(a, b)$ are *in related* $r$ in $\mathcal{K}$ iff every model of $\mathcal{K}$ is a model of $(a, b) : r$

Derived tasks:

- **retrieval:** Given a concept $C$ and an Abox $\mathcal{A}$ find all individuals $a$ such that $\mathcal{K} \models a : C$
- **realization:** Given an individual $a$ and a set of concepts, find the most specific concept $C$ such that $\mathcal{K} \models a : C$. 
Reduction of ABox reasoning tasks

All ABox problems in \( \mathcal{ALC} \) are reducible to ABox consistency. For KB \( \mathcal{K} = (\mathcal{T}, \mathcal{A}) \):

- \( a \) is an instance of \( C \) in \( \mathcal{K} \) \( \iff \mathcal{A} \cup \{ a : \neg C \} \) is inconsistent w.r.t. \( \mathcal{T} \).

  Proof: \( a \) is an instance of \( C \) in \( \mathcal{K} \)
  \( \iff \) for every model \( \mathcal{I} \) of \( \mathcal{K} \) it holds that \( a^\mathcal{I} \in C^\mathcal{I} \)
  \( \iff \) there is no model \( \mathcal{I} \) of \( \mathcal{K} \) s.t. \( a^\mathcal{I} \in (\neg C')^\mathcal{I} \)
  \( \iff \) there is no model \( \mathcal{I} \) of both \( \mathcal{A} \cup \{ a : \neg C \} \) and \( \mathcal{T} \)
  \( \iff \mathcal{A} \cup \{ a : \neg C' \} \) is inconsistent w.r.t. \( \mathcal{T} \).

- \((a, b)\) are in relation \( r \) \( \iff (a, b) : r \in \mathcal{A}. \)

- retrieval and realization equivalent to a finite number of instance checking and subsumption tasks.
Reduction of reasoning tasks

...and finally:

- \( C \) is \textit{satisfiable} w.r.t. \( \mathcal{T} \) \( \iff \) \( \mathcal{A} = \{ a : C \} \) is \textit{consistent} w.r.t. \( \mathcal{T} \), for a fresh individual name \( a \)

Proof:
- \( C \) is satisfiable w.r.t. \( \mathcal{T} \)
  \( \iff \) there is a model \( \mathcal{I} \) of \( \mathcal{T} \) such that \( C^\mathcal{I} \neq \emptyset \)
  \( \iff \) there is at least one instance of \( C \) in \( \mathcal{I} \) — name it \( a \)
  \( \iff \) there is a model of \( \mathcal{T} \) which satisfies assertion \( a : C \)
  \( \iff \) there is a model of \( \mathcal{T} \) which is a model of
    the ABox \( \mathcal{A} = \{ a : C \} \)
  \( \iff \) \( \mathcal{A} = \{ a : C \} \) is consistent w.r.t. \( \mathcal{T} \), for a fresh name \( a \).

Hence:

- All reasoning tasks in \( \mathcal{ALC} \) can be reduced to a single task of checking ABox consistency w.r.t. TBox.
- The complexity of ABox consistency checking cannot be lower than that of the other tasks.
Summary

- A DL knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ consists of the TBox (terminology) $\mathcal{T}$ and the ABox (assertions) $\mathcal{A}$.
- Axioms of a KB restrict the possible models.
- The reasoning tasks in $\mathcal{ALC}$ for TBoxes and ABoxes can be reduced to checking ABox consistency w.r.t. TBox.

Next:

- Tableau algorithm for reasoning in $\mathcal{ALC}$.