Relativizing Concept Descriptions to Comparison Classes in $\mathcal{ALC}$

Szymon Klarman and Stefan Schlobach

Department of Artificial Intelligence, Vrije Universiteit Amsterdam,
De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
sklarman@few.vu.nl, schlobac@few.vu.nl

Abstract. Context-sensitivity has been for long a subject of study in linguistics, logic and computer science. Recently the problem of representing and reasoning with contextual knowledge has been brought up in the research on the Semantic Web. In this paper we introduce a conservative extension to the Description Logic $\mathcal{ALC}$ which supports representation of ontologies containing *relative terms*, such as 'big' or 'tall', whose meaning depends on the reference to a particular *comparison class* (context). We specify the syntax and semantics of the language and present a sound and complete tableau decision procedure.

1 Introduction

It is a commonplace observation that the same expressions might have different meanings when used in different contexts. A trivial example might be that of the concept THE_BIGGEST. Figure 1 presents three snapshots of the same knowledge base that focus on different parts of the domain. The extension of the concept visibly varies across the three takes. Intuitively, there seem to be no contradiction in the fact that individual Moscow is an instance of THE_BIGGEST, when considered in the context of European cities, an instance of $\neg$THE_BIGGEST,

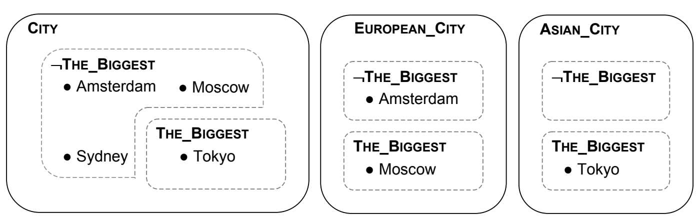

Fig. 1. Example of a relative concept THE_BIGGEST.

when contrasted with all cities, and finally, not belonging to any of these when the focus is only on Asian cities. Natural language users resolve such superficial incoherencies simply by recognizing that certain terms, call them *relative*[1], such as The_Biggest, acquire definite meanings only when put in the context of other denoting expressions — in this case, expressions denoting specific collections of objects, so-called *comparison classes*, with respect to which the relative terms are used [2,3,4].

The problem of context-sensitivity has been for a long time a subject of studies in linguistics, logic and even computer science. Recently, it has been also encountered in the research on the Semantic Web [5,6], where the need for representing and reasoning with imperfect information becomes ever more pressing. Relativity of meaning appears as one of common types of such imperfection. Alas, *Description Logics* (DLs), which form the foundation of the Web Ontology Language [7], the basic knowledge representation formalism on the Semantic Web, were originally developed for modeling crisp, static and unambiguous knowledge, and as such, are incapable of handling the task seamlessly. Consequently, it has become highly desirable to look for more expressive, ideally backward compatible languages to meet the new application requirements [8,9].

In this paper we define a simple, conservative extension to the DL $\mathcal{ALC}$, which is intended for representation of context-sensitive terminologies, where by *contexts* we understand specifically comparison classes with respect to which the relative terms acquire precise meanings. In the following section we formally define the language, next we present a tableau calculus for deciding satisfiability, and finally, in the last two sections, we shortly position our work in a broader perspective and conclude the presentation.

2 Representation Language

The language $\mathcal{C}_{\mathcal{ALC}}$, introduced in this section, extends the basic $\mathcal{ALC}$ logic with two modal-like operators, enabling construction of contextualized concept descriptions, which internalize the use of comparison classes in the language. The classes are denoted by means of arbitrary concept descriptions. Formally, the novel feature of $\mathcal{C}_{\mathcal{ALC}}$ is founded on an extra modal dimension incorporated into DL interpretations, whose possible states are represented by subsets of the object domain.

In the following subsections we first shortly recap the basic notions concerning DLs and then give a detailed account of the syntax and semantics of $\mathcal{C}_{\mathcal{ALC}}$.

2.1 Description Logic $\mathcal{ALC}$

A DL language is specified by a signature $\Sigma = (N_I, N_C, N_R)$, where N_I is a set of *individual names*, N_C a set of *concept names*, and N_R a set of *role names*, and a set of logical operators enabling construction of complex formulas [10].

[1] Philosophy of language qualifies such terms generically as *syncategorematic*. More precisely, a term is syncategorematic if it does not form a denoting expression by itself. See e.g. [1].

The DL $\mathcal{ALC}$ sanctions concept descriptions defined by means of concept names (atomic concepts), special symbols $\top, \bot$ and the following concept constructors:

$$C, D, r \rightarrow \neg C \mid C \sqcap D \mid C \sqcup D \mid \exists r.C \mid \forall r.C$$

A knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ in $\mathcal{ALC}$, consists of the terminological and the assertional component. A (general) TBox $\mathcal{T}$ contains concept inclusion axioms $C \sqsubseteq D$ (abbreviated to $C \equiv D$ whenever $C \sqsubseteq D$ and $D \sqsubseteq C$) for arbitrary concept descriptions C and D. An ABox $\mathcal{A}$ contains axioms of possibly two forms: concept assertions $C(a)$ and role assertions $r(a, b)$, where a, b are individual names, C a concept, and r a role.

The semantics is defined in terms of an *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty *domain* of individuals, and $\cdot^{\mathcal{I}}$ is an *interpretation function*, which specifies the meaning of the vocabulary by mapping every $a \in N_I$ to an element of $\Delta^{\mathcal{I}}$, every $C \in N_C$ to a subset of $\Delta^{\mathcal{I}}$ and every $r \in N_R$ to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The function is inductively extended over complex terms in a usual way, according to the fixed semantics of the logical operators. An interpretation $\mathcal{I}$ *satisfies* an axiom in either of the following cases:

- $\mathcal{I} \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
- $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$
- $\mathcal{I} \models r(a, b)$ iff $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in r^{\mathcal{I}}$

Finally, $\mathcal{I}$ is said to be a *model* of a DL knowledge base, i.e. it makes the knowledge base true, if and only if it satisfies all its axioms.

2.2 Description Logic $\mathcal{C}_{\mathcal{ALC}}$

The syntax of $\mathcal{ALC}$ is extended in $\mathcal{C}_{\mathcal{ALC}}$ with two additional concept constructors, based on modal-like operators $\langle \cdot \rangle$ and $[\cdot]$:

$$C, D \rightarrow \langle D \rangle C \mid [D]C$$

A *contextualized concept description* consists of a *relative concept* C and a specified *comparison class* D, which co-determines the meaning of C. Intuitively, $\langle D \rangle C$ denotes all objects which are C as considered in the context of all objects which are D, whereas $[D]C$ denotes all objects which are either not D or otherwise (like in the former case) are C as considered in the context of all objects which are D. For instance, $\langle$City$\rangle$The_Biggest describes the individuals that are the biggest as considered in the context of (all and only) cities, while [European_City]$\neg$The_Biggest describes all those individuals that are either not European cities, or if they are, they are not the biggest ones in that context. Other than that $\mathcal{C}_{\mathcal{ALC}}$ does not differ from $\mathcal{ALC}$ on the syntactic level.

More interesting changes appear in the semantics of the language, which is essentially augmented with an additional modal dimension, with possible states — comparison classes — corresponding to subsets of the (global) domain of interpretation, and the accessibility relation corresponding to the $\supseteq$-ordering of those subsets. In each state the vocabulary (of the relevant part) of the language is interpreted independently from the others. Definition 1 introduces the notion of *context structure* which characterizes an interpretation of a $\mathcal{C}_{\mathcal{ALC}}$ language.

Definition 1. *A **context structure** for a set of languages $\{\mathcal{L}_w\}_{w\in W}$ is a tuple $\mathcal{C} = \langle W, \lhd, \Delta, \{\mathcal{I}_w\}_{w\in W}\rangle$, where:*

- *$W \subseteq \wp(\Delta)$ is a set of **comparison classes**, with $\Delta \in W$ and $\emptyset \notin W$,*
- *$\lhd \subseteq W \times W$ is an **accessibility relation**, s.t. $w \lhd v$ iff $v \subseteq w$*
- *Δ is a **global domain** of interpretation,*
- *$\mathcal{I}_w = (\Delta^{\mathcal{I}_w}, \cdot^{\mathcal{I}_w})$ is an **interpretation** of $\mathcal{L}_w$ with respect to w:*
  - *$\Delta^{\mathcal{I}_w} = w$ is a non-empty domain of individuals,*
  - *$\cdot^{\mathcal{I}_w}$ is an interpretation function defined as usual.*

Instead of speaking of one language $\mathcal{L}$, in many cases it is more convenient to refer to particular sublanguages $\{\mathcal{L}_w\}_{w\in W}$, which are based on the parts of the vocabulary of $\mathcal{L}$ deemed meaningful in particular contexts.[2] We assume that individual names are interpreted rigidly, i.e. for every $a \in N_I$ and every $w, v \in W$, such that a belongs to the vocabulary of $\mathcal{L}_w$ and $\mathcal{L}_v$, the context structure has to satisfy $a^{\mathcal{I}_w} = a^{\mathcal{I}_v}$. It is easy to observe that $\lhd$ imposes a partial order (reflexive, asymmetric and transitive) on the set of contexts, with the root $\hat{w} = \Delta \in W$ as its least element. Thus context structures correspond to rooted, partially ordered Kripke frames.

Given a context structure $\mathcal{C} = \langle W, \lhd, \Delta, \{\mathcal{I}_w\}_{w\in W}\rangle$ we can now properly define the semantics of contextualized concept descriptions.

$$(\langle D\rangle C)^{\mathcal{I}_w} = \{x \in \Delta^{\mathcal{I}_w} \mid \exists w \lhd v,\ \Delta^{\mathcal{I}_v} = D^{\mathcal{I}_w} :\ x \in C^{\mathcal{I}_v}\}$$
$$([D]C)^{\mathcal{I}_w} = \{x \in \Delta^{\mathcal{I}_w} \mid \forall w \lhd v,\ \Delta^{\mathcal{I}_v} = D^{\mathcal{I}_w} : x \in \Delta^{\mathcal{I}_v} \text{ implies } x \in C^{\mathcal{I}_v}\}$$

As usual the operators can be defined in terms of their dual counterparts, i.e. $[D]C = \neg\langle D\rangle\neg C$ and $\langle D\rangle C = \neg[D]\neg C$. Furthermore, another less common relationship between the operators can also be derived, namely: $[D]C = \neg D \sqcup \langle D\rangle C$ and $\langle D\rangle C = D \sqcap [D]C$.

As expected, the notion of satisfaction in $\mathcal{C}_{\mathcal{ALC}}$ is relativized to context structure and a particular comparison class in that structure. A context structure $\mathcal{C}$ is a model of a knowledge base if and only if all the axioms in that knowledge base are satisfied at the root of $\mathcal{C}$.

- $\mathcal{C}, \hat{w} \models C \sqsubseteq D$ iff $C^{\mathcal{I}_{\hat{w}}} \subseteq D^{\mathcal{I}_{\hat{w}}}$
- $\mathcal{C}, \hat{w} \models C(a)$ iff $a^{\mathcal{I}_{\hat{w}}} \in C^{\mathcal{I}_{\hat{w}}}$
- $\mathcal{C}, \hat{w} \models r(a, b)$ iff $\langle a^{\mathcal{I}_{\hat{w}}}, b^{\mathcal{I}_{\hat{w}}}\rangle \in r^{\mathcal{I}_{\hat{w}}}$

Note, that we interpret all axioms only at the roots of models. It follows that both syntactically and semantically $\mathcal{C}_{\mathcal{ALC}}$ is a conservative extension of $\mathcal{ALC}$, i.e. every satisfiable $\mathcal{ALC}$ knowledge base is a satisfiable $\mathcal{C}_{\mathcal{ALC}}$ knowledge base.

[2] For instance, the name Moscow is not meaningful when the focus is only on the Asian cities. Similarly, some concepts can be applicable only with respect to particular types of objects, e.g. natural numbers can be neither Hot nor ¬Hot.

3 Tableau Calculus

The tableau calculus for $\mathcal{C}_{\mathcal{ALC}}$, presented in this section, is an extension of the well-known procedures for $\mathcal{ALC}$ [11,12]. The proof of satisfiability of a formula ϑ is a process of finding a complete and clash-free *constraint system* for ϑ (a set of logical constraints) by means of tableau rules. If such a system exists then ϑ is satisfiable. The constraint systems are constructed by iterative application of inference rules to the constraints in the system.

Apart from terms representing domain objects $\Lambda_I = N_I \cup \{x, y, \ldots\}$, the calculus involves a set of *context labels* $\Lambda_C = \{\gamma, \delta, \ldots\}$ used for marking comparison classes, where each label is a finite sequence of concept descriptions separated with a vertical line: $\gamma = C_1 \mid C_2 \mid \ldots \mid C_n$. The empty label $\epsilon \in \Lambda_C$ refers to the root of the context structure. The labels can be rendered into $\mathcal{C}_{\mathcal{ALC}}$ by means of the function $p : \Lambda_C \mapsto \mathcal{L}$, such that for every $\gamma = C_1 \mid \ldots \mid C_{n-1} \mid C_n \in \Lambda_C$:

$$p(\gamma) = \langle C_1 \rangle \ldots \langle C_{n-1} \rangle C_n \ ; \qquad p(\epsilon) = \top$$

$\Rightarrow_\sqcap$ **if** $\gamma : (C \sqcap D)(x) \in S$
then set $S' := S \cup \{\gamma : C(x), \gamma : D(x)\}$

$\Rightarrow_\sqcup$ **if** $\gamma : (C \sqcup D)(x) \in S$
then set $S' := S \cup \{\gamma : C(x)\}$ **or** $S' := S \cup \{\gamma : D(x)\}$

$\Rightarrow_\exists$ **if** $\gamma : (\exists r.C)(x) \in S$ **and** x is not blocked in γ
then set $S' := S \cup \{\gamma : r(x, y), \gamma : C(y)\}$, for a new $\ll$-minimal y

$\Rightarrow_\forall$ **if** $\gamma : (\forall r.C)(x) \in S$ **and** $\gamma : r(x, y) \in S$
then set $S' := S \cup \{\gamma : C(y)\}$

$\Rightarrow_\equiv$ **if** $\epsilon : (\top \equiv C) \in S$ **and** $\epsilon : \phi(x) \in S$
then set $S' := S \cup \{\epsilon : C(x)\}$

$\Rightarrow_{\not\equiv}$ **if** $\epsilon : (C \not\equiv D) \in S$
then set $S' := S \cup \{\epsilon : C(x), \neg D(x)\}$
or $S' := S \cup \{\epsilon : \neg C(x), D(x)\}$, for a a new $\ll$-minimal x

$\Rightarrow_{\mathbf{B}}$ **if** $\{C \mid \gamma : C(y) \in S\} \subseteq \{C \mid \gamma : C(x) \in S\}$
for a fixed γ and $y \ll x$ **then** mark y as blocked in γ by x

Table 1. $\mathcal{ALC}$ tableau rules.

A tableau proof for $\vartheta = \bigwedge_i \vartheta_i$, where every ϑ_i is a DL axiom, is initiated by setting a constraint system containing $\epsilon : \vartheta_i$ for all i. At every stage of the procedure, the system contains only elements of the form: (1) $\gamma : C(x)$; (2) $\gamma : r(x, y)$; (3) $\epsilon : \top \equiv C$; (4) $\epsilon : C \not\equiv D$; for some $\gamma \in \Lambda_C$, $x, y \in \Lambda_I$ and concept/role descriptions C, D, r. Note that TBox axioms can be rewritten in a usual manner into a single constraint $\epsilon : \top \equiv \bigsqcap_{C \sqsubseteq D \in \mathcal{T}} \neg C \sqcup D$. We moreover assume that all concepts descriptions are given in the Negation Normal Form, and that all occurrences of $[D]C$ are replaced with the equisatisfiable $\neg D \sqcup \langle D\rangle C$.

The standard $\mathcal{ALC}$ inference rules are restated in Table 1. As usual we assume a well-ordering $\ll$ of the individual variables used in a proof for a proper application of the blocking mechanism. We always require that applications of the $\Rightarrow_\exists$ rule are deferred until no other rules apply. We say that a system contains a clash if for some γ, x and A it contains both $\gamma : A(x)$ and $\gamma : \neg A(x)$. Besides

$\Rightarrow_{\langle\cdot\rangle}$	**if** $\gamma : \langle C\rangle D(x) \in S$ **then set** $S' := S \cup \{\gamma \mid C : D(x)\}$
$\Rightarrow_{\supset}$	**if** $\gamma \mid C : \phi(x) \in S$ **then set** $S' := S \cup \{\gamma : C(x)\}$
$\Rightarrow_{\not\equiv}$	**if** $\gamma : C(x) \in S$ **and** $\delta : \neg C(x) \in S$ **then set** $S' := S \cup \{\epsilon : p(\gamma) \not\equiv p(\delta)\}$

Table 2. $\mathcal{C}_{\mathcal{ALC}}$ tableau rules.

the $\mathcal{ALC}$ component, the inference engine comprises three additional rules, presented in Table 2. The meaning of the $\Rightarrow_{\langle\cdot\rangle}$ rule is straightforward: it introduces a relative concept assertion within the scope of a newly generated context label, this way marking a transition of the proof into a different comparison class. The two remaining rules require some more comment. The $\Rightarrow_{\supset}$ rule accounts for the

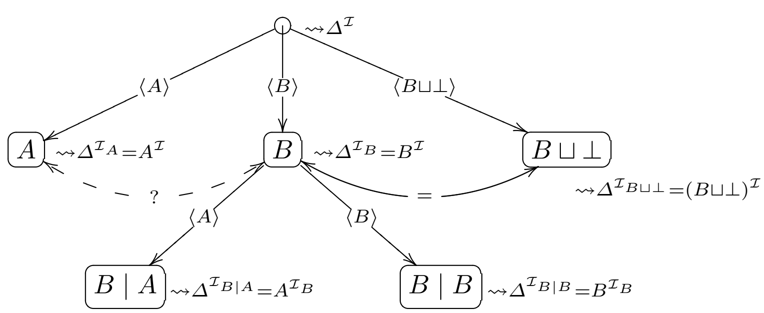

Fig. 2. A tree of context labels and the underlying context structure.

fact that the accessibility relation between comparison classes has to follow their $\supseteq$-ordering. Hence all individual terms introduced in a proof within the scope of a particular context label have to be simultaneously represented within the scope of its predecessor, and moreover they have to occur there as instances of the concept which denotes the successor comparison class. The $\Rightarrow_{\supset}$ rule ensures that exactly these conditions are satisfied. The $\Rightarrow_{\neq}$ rule resolves synonymity between certain context labels. Observe, that the labeling generated by $\Rightarrow_{\langle\cdot\rangle}$ is tree-shaped (see Figure 2), while the $\lhd$ ordering of the comparison classes does not have to be such in principle. In fact, different context labels (or rather their p-translations) might denote exactly the same comparison classes (e.g. consider operators $\langle B\rangle$ and $\langle B \sqcup \bot\rangle$ in the picture). This is not a problem as long as there is no potential clash between the assertions occurring within the scope of different labels. In case there is such a possibility, the rule ensures that the labels indeed represent non-equivalent comparison classes.

It can be shown that the basic results of soundness, termination and completeness hold for the calculus (the proofs are included in the Appendix):

Lemma 1 (Soundness). *Let S be a complete clash-free constraint system for ϑ. Then ϑ is satisfiable.*

Lemma 2 (Termination). *There is no infinite sequence of inference steps via the tableau rules.*

Lemma 3 (Completeness). *If ϑ is satisfiable then there exists a complete clash-free constraint system for ϑ.*

4 Relative Terminologies — Example

For a small example of a $\mathcal{C}_{\mathcal{ALC}}$ knowledge base we will formalize part of Figure 1 depicted in the introduction. Consider knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where TBox and ABox are defined as follows:

$$
\begin{aligned}
\mathcal{T} = \{\ & (1)\ \textsc{City} \equiv \textsc{European\_City} \sqcup \textsc{Asian\_City},\\
& (2)\ \textsc{European\_City} \sqcap \textsc{Asian\_City} \sqsubseteq \bot,\\
& (3)\ \langle\textsc{City}\rangle\textsc{The\_Biggest} \equiv \langle\textsc{Asian\_City}\rangle\textsc{The\_Biggest}\ \}\\
\mathcal{A} = \{\ & (4)\ \langle\textsc{City}\rangle\textsc{The\_Biggest}(\mathsf{Tokyo}),\\
& (5)\ \langle\textsc{European\_City}\rangle\textsc{The\_Biggest}(\mathsf{Moscow})\ \}
\end{aligned}
$$

The TBox states that every city is either a European or an Asian city, and that, in fact, the two classes are disjoint. The third axiom ensures that the concept THE_BIGGEST has the same instances in the context of all cities, and in the context of Asian cities. The axioms in the ABox assert that Tokyo is the biggest among all cities, whereas Moscow is the biggest in the context of European cities. Given this setup it can be shown, for instance, that the following entailments hold:

$$
\begin{aligned}
&\mathcal{K} \models \langle\textsc{City} \sqcap \neg\textsc{European\_City}\rangle\textsc{The\_Biggest}(\mathsf{Tokyo})\\
&\mathcal{K} \models \langle\textsc{City}\rangle\neg\textsc{The\_Biggest}(\mathsf{Moscow})
\end{aligned}
$$

Due to limited space we will not present full tableau proofs and resort only to informal arguments. The validity of the first entailment rests on the fact that according to the TBox Asian cities are exactly those that are non-European cities (from 1). The comparison class denoted by $\textsc{Asian\_City}$ is therefore equivalent to that described by $\textsc{City} \sqcap \neg\textsc{European\_City}$, and so the two descriptions represent in fact the same state in the context structure. Consequently, since Tokyo is an instance of $\textsc{The\_Biggest}$ in the former context (from 4 and 3), it has to be such also in the latter.

To see that the second entailment follows as well, assume on the contrary that $\neg\langle\textsc{City}\rangle\neg\textsc{The\_Biggest}(\mathsf{Moscow})$, i.e. $\neg\textsc{City} \sqcup \langle\textsc{City}\rangle\textsc{The\_Biggest}(\mathsf{Moscow})$. Clearly Moscow is a city (from 5 and 1), hence it would have to be an instance of $\textsc{The\_Biggest}$ in the context of all cities, and consequently, in the context of Asian cities (from 3). But this would mean that Moscow is an Asian city, which is not true (from 2 and 5).

Note that, as intended, there is no contradiction between the fact that Moscow is an instance of $\textsc{The\_Biggest}$ in the context of European cities and an instance of $\neg\textsc{The\_Biggest}$ in the context of all cities. Finally we can define a context structure $\mathcal{C} = \langle W, \lhd, \Delta, \{\mathcal{I}_w\}_{w\in W}\rangle$ which models $\mathcal{K}$. We pose:

$$
\begin{aligned}
W &= \{\hat{w} = \{\mathsf{Moscow}, \mathsf{Tokyo}\}, w_1 = \{\mathsf{Moscow}\}, w_2 = \{\mathsf{Tokyo}\}\} \\
\lhd &= \{\langle\hat{w}, w_1\rangle, \langle\hat{w}, w_2\rangle\} \\
\Delta = \Delta^{\mathcal{I}_{\hat{w}}} &= \{\mathsf{Moscow}, \mathsf{Tokyo}\} \\
\textsc{City}^{\mathcal{I}_{\hat{w}}} &= \{\mathsf{Moscow}, \mathsf{Tokyo}\} \\
\textsc{European\_City}^{\mathcal{I}_{\hat{w}}} &= \{\mathsf{Moscow}\} \\
\textsc{Asian\_City}^{\mathcal{I}_{\hat{w}}} &= \{\mathsf{Tokyo}\} \\
\textsc{The\_Biggest}^{\mathcal{I}_{\hat{w}}} &= \{\mathsf{Tokyo}\} \\
\mathsf{Moscow}^{\mathcal{I}_{\hat{w}}} &= \mathsf{Moscow} \\
\mathsf{Tokyo}^{\mathcal{I}_{\hat{w}}} &= \mathsf{Tokyo} \\
\Delta^{\mathcal{I}_{w_1}} &= \{\mathsf{Moscow}\} \\
\textsc{The\_Biggest}^{\mathcal{I}_{w_1}} &= \{\mathsf{Moscow}\} \\
\mathsf{Moscow}^{\mathcal{I}_{w_1}} &= \mathsf{Moscow} \\
\Delta^{\mathcal{I}_{w_2}} &= \{\mathsf{Tokyo}\} \\
\textsc{The\_Biggest}^{\mathcal{I}_{w_2}} &= \{\mathsf{Tokyo}\} \\
\mathsf{Tokyo}^{\mathcal{I}_{w_2}} &= \mathsf{Tokyo}
\end{aligned}
$$

5 Related Work

The logic discussed in the previous sections can be seen as a special case of *multi-dimensional DLs* [13,14], and more generally, as an instance of *multi-dimensional modal logics* [15,12], in which next to the standard object dimension we introduce a second one, referring to the subsets of the object domain as the possible states in the model. The scope of multi-dimensionality involved here, however, is very limited, thus discharging a number of computational problems, which otherwise are inherent to richer multi-dimensional formalisms. Notably, we were able to define a terminating decision procedure without resorting to certain advanced techniques such as based on *quasimodels* [16,12].

In general, the problem of representing and reasoning with contextual knowledge, in particular in relation to DL-like knowledge bases, has been quite broadly studied in the literature, for instance in [6,17,18,19,20]. However, the vast majority of authors considers the notion of context on a very abstract level, merely as a specific (limited) view on the application domain, without explicating the formal character of that specificity. In particular there has been no attempt of formalizing contexts as comparison classes in DL. On the other hand, the general semantic intuition of introducing an additional modal dimension, in an explicit (e.g. by context indexing [18]) or an implicit (e.g. by reference to subsets of possible models of knowledge base [19]) manner, remains the same as in our approach.

Finally, from a different perspective, $\mathcal{C}_{\mathcal{ALC}}$ shares also some significant similarities with *dynamic epistemic logics*, and in particular, with the *public announcement logic* (PAL) [21], which studies the dynamics of information flow in epistemic models. Interestingly, the two special operators used in $\mathcal{C}_{\mathcal{ALC}}$ can be to some extent interpreted as public announcements (in the sense used in PAL), whose role is to reduce the description (epistemic) model to only those individuals (epistemic states) that satisfy given description (formula). Unlike in PAL, however, we allow for much deeper revisions of the models, involving also the interpretation function, e.g. it is possible that after contextualizing the representation by $\langle C \rangle$ there are no individuals that are C, simply because C gets essentially reinterpreted in the accessed context. For that reason it is also not possible to reduce reasoning in $\mathcal{C}_{\mathcal{ALC}}$ to the PAL case, for which there exist tableau proof procedures, e.g. [22].

6 Conclusions and Future Work

Providing a sound formal account of context-sensitivity and related phenomena, which abound in the real-life knowledge representation and reasoning scenarios, is a longstanding challenge in Artificial Intelligence, and quite recently, also in the research on the Semantic Web. In this paper we have addressed a very specific form of that problem, namely, representation of *relative terms*, whose meaning depends on the selection of comparison classes, to which the terms are applied. For such tasks we have proposed language $\mathcal{C}_{\mathcal{ALC}}$, a simple extension of the DL $\mathcal{ALC}$, which by utilizing an additional modal dimension introduced into standard $\mathcal{ALC}$ models allows for a roughly independent interpretation of the vocabulary of the language in the context of different comparison classes. This way the resulting reasoning regime complies much closer to the intuitions associated with the use of relative terms in the natural language, e.g. it does not allow for inferring contradiction based on existence of complementary statements about an object, as long as the statements apply to different comparison classes.

Admittedly, the scope of the proposal is very limited and in no way can it pretend to have solved the problem of context-sensitivity in the DL-based representations of knowledge. Nevertheless, we have showed that by a careful use of supplementary modal dimensions one can obtain extra expressive power, which on the one hand is sufficient to handle certain interesting representation problems, while on the other does not require deep revisions on the syntactic, semantic nor, most importantly, the proof-theoretic side of the basic DL paradigm. Our belief is that in a similar manner, aspects of *multi-dimensionality* can offer convenient formal means for addressing other phenomena related to imperfect knowledge, such as *uncertainty* or *vagueness*, which currently are approached on the grounds of formalisms involving a thorough reconstruction of the semantics and the proof theory of DLs, e.g. probabilistic, possibilistic or fuzzy DLs [8].

In the future work on $\mathcal{C}_{\mathcal{ALC}}$ we want to focus on defining suitable mechanisms of support for both relative and absolute (rigid) terms, preferably by distinguishing between local (contextualized) and global (applicable to all contexts) terminological constraints, so that some parts of vocabulary can be rendered context-independent. In a broader perspective, we intend to investigate ways of extending the approach to other types of context-sensitivity as well as to other aspects of imperfect knowledge representations.

Acknowledgements The first author is funded by a Hewlett-Packard Labs Innovation Research Program.

References

1. Quine, W.V.O.: Word and Object. MIT Press: Cambridge (1960)
2. Shapiro, S.: Vagueness in Context. Oxford University Press (2006)
3. van Rooij, R.: Vagueness and Linguistics. In: The Vagueness Handbook. Springer (to appear)
4. Gaio, S.: Granular Models for Vague Predicates. In Eschenbach, C., Grüninger, M., eds.: Formal Ontology in Information Systems. Proceedings of the Fifth International Conference (FOIS 2008), IOS Press, Amsterdam (2008) 253–265
5. Caliusco, M.L., Galli, M.R., Chiotti, O.: Contextual ontologies for the semantic web — an enabling technology. In: LA-WEB '05: Proceedings of the Third Latin American Web Congress, IEEE Computer Society (2005)
6. Benslimane, D., Arara, A., Falquet, G., Maamar, Z., Thiran, P., Gargouri, F.: Contextual ontologies: Motivations, challenges, and solutions. In: Proceedings of the Advances in Information Systems Conference, Izmir. (2006)
7. Horrocks, I., Patel-schneider, P.F., Harmelen, F.V.: From SHIQ and RDF to OWL: The making of a Web Ontology Language. Journal of Web Semantics **1** (2003) 2003
8. Lukasiewicz, T., Straccia, U.: Managing uncertainty and vagueness in description logics for the semantic web. Web Semantics **6**(4) (2008) 291–308
9. Laskey, K.J., Laskey, K.B., Mason, G., Costa, P.C.G., Mason, G., Kokar, M.M., Martin, T., Lukasiewicz, T.: Uncertainty Reasoning for the World Wide Web, W3C Incubator Group report 31 march 2008. `http://www.w3.org/2005/Incubator/urw3/XGR-urw3-20080331/` (2008)
10. Baader, F., Calvanese, D., Mcguinness, D.L., Nardi, D., Patel-Schneider, P.F.: The description logic handbook: theory, implementation, and applications. Cambridge University Press (2003)

11. Baader, F., Sattler, U.: An overview of tableau algorithms for description logics. Studia Logica **69** (2001) 5–40
12. Kurucz, A., Wolter, F., Zakharyaschev, M., Gabbay, D.M.: Many-Dimensional Modal Logics: Theory and Applications. Number 148 in Studies in Logic and the Foundations of Mathematics. Elsevier (2003)
13. Wolter, F., Zakharyaschev, M.: Multi-dimensional description logics. In: IJCAI '99: Proceedings of the Sixteenth International Joint Conference on Artificial Intelligence, San Francisco, CA, USA, Morgan Kaufmann Publishers Inc. (1999) 104–109
14. Wolter, F., Zakharyaschev, M.: Modal description logics: modalizing roles. Fundamenta Informaticae **39**(4) (1999) 411–438
15. Marx, M., Venema, Y.: Multi Dimensional Modal Logic. Number 4 in Applied Logic Series. Kluwer Academic Publishers (1997)
16. Wolter, F., Zakharyaschev, M.: Satisfiability problem in description logics with modal operators. In: In Proceedings of the Sixth Conference on Principles of Knowledge Representation and Reasoning, Morgan Kaufman (1998) 512–523
17. Giunchiglia, F., , Ghidini, C.: Local models semantics, or contextual reasoning = locality + compatibility. Artificial Intelligence **127** (2001)
18. Bouquet, P., Giunchiglia, F., van Harmelen, F., Serafini, L., Stuckenschmidt, H.: C-OWL: Contextualizing ontologies. In Fensel, D., Sycara, K.P., Mylopoulos, J., eds.: International Semantic Web Conference. Volume 2870 of Lecture Notes in Computer Science., Springer (2003) 164–179
19. Grossi, D.: Desigining Invisible Handcuffs. Formal Investigations in Institutions and Organizations for Multi-Agent Systems. Siks dissertation series 2007-16, Utrecht University (2007)
20. Goczyla, K., Waloszek, W., Waloszek, A.: Contextualization of a DL knowledge base. In Calvanese, D., Franconi, E., Haarslev, V., Lembo, D., Motik, B., Turhan, A.Y., Tessaris, S., eds.: The Proceedings of the International Workshop on Description Logics (DL-2007). (2007)
21. van Ditmarsch, H., van der Hoek, W., Kooi, B.: Dynamic Epistemic Logic. Synthese Library. Springer (2007)
22. Balbiani, P., Ditmarsch, H., Herzig, A., Lima, T.: A tableau method for public announcement logics. In: TABLEAUX '07: Proceedings of the 16th international conference on Automated Reasoning with Analytic Tableaux and Related Methods, Berlin, Heidelberg, Springer-Verlag (2007) 43–59

Appendix

Below we present proofs of the lemmas from Section 3. The structure and notation follows closely the presentation of the corresponding results for $\mathcal{ALC}$ in [12]. In particular we use $con(\vartheta)$ to denote all subconcepts of ϑ and $obj(\vartheta)$ for all individual names occurring in the formula.

Proof of Lemma 1 We use S as 'a guide' to show that there exists a model for ϑ. First, we define a structure $\mathcal{C} = \langle W, \lhd, \Delta, \{\mathcal{I}_w\}_{w\in W}\rangle$ as follows:

- $w_\gamma \in W$ for every γ occurring in S, where $w_\gamma = \Delta^{\mathcal{I}_{w_\gamma}}$ comprises all terms x such that $\gamma : \phi(x) \in S$
- $\gamma \lhd \delta$ for every γ and $\delta = \gamma \mid C$ occurring in S
- $x \in A^{\mathcal{I}_{w_\gamma}}$ for every γ, x and A such that $\gamma : A(x) \in S$
- $a^{\mathcal{I}_{w_\gamma}} = a$ for every γ and $a \in obj(\vartheta)$ such that $\gamma : \phi(a) \in S$
- $r^{\mathcal{I}_{w_\gamma}}(x, y)$ for every γ, x, y and r such that $\gamma : r(x, y) \in S$ (or $\gamma : r(z, y) \in S$ in case x is blocked in γ by z)

Next we prove the following claim.

Proposition 1. *For every $w_\gamma \in W$, concept $C \in con(\vartheta)$ and $x \in \Delta^{\mathcal{I}_{w_\gamma}}$, if $\gamma : C(x) \in S$ then $x \in C^{\mathcal{I}_{w_\gamma}}$.*

Proof. We prove the claim by induction on the syntactic form of C.

- For atomic concepts the claim follows directly from the definition of $\mathcal{C}$.
- Let $C = \neg A$ for some atomic concept A and suppose $\gamma : \neg A(x) \in S$ for any γ and x. Notice that there can be no $\gamma : A(x) \in S$, as S is clash-free. Hence $x \notin A^{\mathcal{I}_{w_\gamma}}$ and $x \in (\neg A)^{\mathcal{I}_{w_\gamma}}$.
- The cases of $C = B \sqcap D$, $C = B \sqcup D$, $C = \exists r.D$ and $C = \forall r.D$ are as in $\mathcal{ALC}$ (see [12]), modulo proper indexing of the interpretation function.
- Let $C = \langle B\rangle D$ and suppose $\gamma : (\langle B\rangle D)(x) \in S$. Then since S is closed under $\Rightarrow_{\langle\cdot\rangle}$, it has to be the case that $\gamma \mid B : D(x)$. But then there exists $w_{\gamma|B} \in W$, such that $w_\gamma \lhd w_{\gamma|B}$ and $x \in D^{\mathcal{I}_{w_{\gamma|B}}}$. Also, since S is closed under $\Rightarrow_{\supset}$, it has to be the case that $\Delta^{\mathcal{I}_{w_{\gamma|B}}} = B^{\mathcal{I}_{w_\gamma}}$. Therefore $x \in (\langle B\rangle D)^{\mathcal{I}_{w_\gamma}}$. ❑

By Proposition 1 we can immediately show the following result:

Proposition 2. *Let $\hat{w}$ be the root of $\mathcal{C}$. Then $\mathcal{C}, \hat{w} \models \vartheta$.*

Proof. Recall that $\vartheta = \bigwedge_i \vartheta_i$ and for all i, $\epsilon : \vartheta_i$ is in the initial constraint system for ϑ with empty context labels. Consider possible syntactic form of any ϑ_i:

- $C(a)$: then by Prop. 1, $a \in C^{\mathcal{I}_{\hat{w}}}$ and hence $\mathcal{C}, \hat{w} \models C(a)$.
- $r(a, b)$: then by definition of $\mathcal{C}$ it holds that $\langle a^{\mathcal{I}_{\hat{w}}}, b^{\mathcal{I}_{\hat{w}}}\rangle \in r^{\mathcal{I}_{\hat{w}}}$ and hence $\mathcal{C}, \hat{w} \models r(a, b)$.
- $\top \equiv C$: then since S is closed under $\Rightarrow_{\equiv}$, it has to be the case that for all $x \in \Delta^{\mathcal{I}_{\hat{w}}}$, $C(x) \in S$ and by Prop. 1, $x \in C^{\mathcal{I}_w}$. Hence $\top^{\mathcal{I}_{\hat{w}}} = C^{\mathcal{I}_{\hat{w}}}$, and consequently $\mathcal{C}, \hat{w} \models \top \equiv C$.
- $C \not\equiv D$: then since S is closed under $\Rightarrow_{\not\equiv}$ there has to be $x \in \Delta^{\mathcal{I}_{\hat{w}}}$ such that $x \in C^{\mathcal{I}_{\hat{w}}}$ and $x \notin D^{\mathcal{I}_{\hat{w}}}$ or vice versa, so that $C^{\mathcal{I}_{\hat{w}}} \neq D^{\mathcal{I}_{\hat{w}}}$ and consequently $\mathcal{C}, \hat{w} \models C \not\equiv D$.

It follows that each conjunct of ϑ is satisfied in $\hat{w}$, and therefore, that $\mathcal{C}, \hat{w} \models \vartheta$. ❑

Finally, observe that the structure $\mathcal{C}$, as defined in the beginning of the proof, might not always satisfy all the constraints posed by Definition 1. Since S is closed under $\Rightarrow_{\supset}$ it has to be the case that $\lhd$ is a subset of the ordering induced by $\supseteq$ on W. In some cases, however, it might be a proper subset, meaning that accessibility relation between the elements of W is not exhaustively represented, and consequently, that several elements of W might actually represent the same comparison classes. We show that there is always a way of obtaining a proper context structure from $\mathcal{C}$, while preserving the properties discussed above.

Proposition 3. *Let $\mathcal{C}'$ be obtained from $\mathcal{C}$ as follows:*

1. *for every $w, v \in W$, if $v \subseteq w$ then set $w \lhd v$.*
2. *for every $w, v \in W$, if $w = v$ then merge w with v, i.e. consistently replace all occurrences of v (both in $\mathcal{C}$ and S) with w and set: $\mathcal{I}^w := (\Delta^{\mathcal{I}_w}, \cdot^{\mathcal{I}_w \uplus \mathcal{I}_v})$, where for any A and r it holds that $A^{\mathcal{I}_w \uplus \mathcal{I}_v} = A^{\mathcal{I}_w} \cup A^{\mathcal{I}_v}$ and $r^{\mathcal{I}_w \uplus \mathcal{I}_v} = r^{\mathcal{I}_w} \cup r^{\mathcal{I}_v}$.*

Then $\mathcal{C}'$ is a context structure, which is a model of ϑ.

Proof. Clearly $\mathcal{C}'$ is a context structure. We only need to show that it is still a model of ϑ. Reconsider the case of atomic concepts and their complements in the proof of Proposition 1. Obviously, for any x, A and $w_\gamma, v_\delta \in W$, if $x \in A^{\mathcal{I}_{w_\gamma}}$ then also $x \in A^{\mathcal{I}_{w_\gamma} \uplus \mathcal{I}_{v_\delta}}$, which means that interpretations of atomic concepts are not affected by the operation of merging equivalent comparison classes. Let now $x \in (\neg A)^{\mathcal{I}_{w_\gamma}}$ and consider merging w_γ with an equivalent $v_\delta \in W$. If $x \in (\neg A)^{\mathcal{I}_{v_\delta}}$ then we also get $x \in (\neg A)^{\mathcal{I}_{w_\gamma} \uplus \mathcal{I}_{v_\delta}}$, which is consistent. Suppose, on the contrary, that $x \in A^{\mathcal{I}_{v_\delta}}$. But this would mean that $\gamma : \neg A(x) \in S$ and $\delta : A(x) \in S$. Since S is closed under $\Rightarrow_{\neq}$ and $\Rightarrow_{\not\equiv}$ it follows that there has to be an individual i such that $i \in \Delta^{\mathcal{I}_{w_\gamma}}$ and $i \notin \Delta^{\mathcal{I}_{v_\delta}}$ or vice versa, and hence w_γ and v_δ are not in fact equivalent, which contradicts the initial assumption. We conclude that the interpretations of complements of atomic concepts are not affected by the merging operation either. The rest of the argument from Proposition 1 follows inductively, and thus Proposition 2 also holds for $\mathcal{C}'$. ❑

This completes the proof of soundness. ❑

Proof of Lemma 2 Consider a formula ϑ in NNF with all occurrences of $[D]C$ transformed into $\neg D \sqcup \langle D \rangle C$. Clearly, there is only a finite number of $\langle \cdot \rangle$ operators used in ϑ, and hence, there can be only a finite number of unique context labels occurring in the tableau proof for ϑ due to application of the $\Rightarrow_{\langle \cdot \rangle}$ rule. Given that, there can be also only finite number of inference steps via the $\Rightarrow_{\neq}$ rule, as well as via the $\Rightarrow_{\supset}$ rule for any individual variable. Note, that other than occurrences of $\langle \cdot \rangle$ ϑ does not contain any symbols from beyond the $\mathcal{ALC}$, hence the only problem for termination is posed by application of the $\Rightarrow_{\exists}$ rule (clearly, upon suspending it there can be always only a finite number of possible inference steps). But given a finite number of context labels it has to be the case that at some point the $\Rightarrow_{\mathrm{B}}$ rule applies, and all $\ll$-minimal individual variables occurring in S are blocked. Hence the tableau procedure for ϑ terminates in a finite number of steps. ❑

Proof of Lemma 3 Let ϑ be satisfiable and let $\mathcal{C} = \langle W, \lhd, \Delta, \{\mathcal{I}_w\}_{w \in W}\rangle$ be a context structure such that $\mathcal{C}, \hat{w} \models \vartheta$. We use $\mathcal{C}$ as an oracle in determining the construction of a complete clash-free constraint system for ϑ. We say that a constraint system S is compatible with $\mathcal{C}$ iff (1) $\mathcal{C} \models \varphi$, for every constraint $\varphi \in S$ and (2) there exist mappings $\pi : \Lambda_C \mapsto W$ and $\sigma : \Lambda_I \mapsto \Delta$, where Λ_C and Λ_I are the context labels and the individual terms occurring in S, which satisfy the conditions:

- $\pi(\epsilon) = \hat{w}$, where $\hat{w}$ is the root of $\mathcal{C}$
- $\langle \pi(\gamma), \pi(\delta)\rangle \in \lhd$ whenever γ and δ occur in S and $\delta = \gamma \mid \ldots$,
- $\sigma(a) = a^{\mathcal{I}_w}$ for every $a \in obj(\vartheta)$ and $w \in W$;
- $\sigma(x) \in \Delta^{\mathcal{I}_{\pi(\gamma)}}$ whenever $\gamma : \phi(x) \in S$;
- $\sigma(x) \in C^{\mathcal{I}_{\pi(\gamma)}}$ whenever $\gamma : C(x) \in S$;
- $\langle \sigma(x), \sigma(y)\rangle \in r^{\mathcal{I}_{\pi(\gamma)}}$ whenever $\gamma : C(x) \in S$.

Let S be a constraint system for ϑ compatible with $\mathcal{C}$. We show that if any of the tableau rules is applicable to S, then it can be applied in such a way that the resulting system S' is still compatible with $\mathcal{C}$.

- The cases of $\Rightarrow_\sqcap$, $\Rightarrow_\sqcup$, $\Rightarrow_\exists$, $\Rightarrow_\forall$, $\Rightarrow_\equiv$ are as in $\mathcal{ALC}$ (see [12]). The mapping π remains unmodified.
- Suppose we apply $\Rightarrow_{\not\equiv}$ to some $C \not\equiv D \in S$. Then we obtain S' by adding either $C(x)$ and $\neg D(x)$ or $\neg C(x)$ and $D(x)$ to S for some fresh $\ll$-minimal variable y. Clearly, one of the two must be satisfied in $\mathcal{C}$. By picking the correct one we meet (1). For (2), we leave π unmodified, and set $\sigma(x) := d$ for some $d \in \Delta$ such that $d \notin C^{\mathcal{I}_{\pi()}} \cap D^{\mathcal{I}_{\pi()}}$.
- Suppose we apply $\Rightarrow_{\langle\cdot\rangle}$ to some $\gamma : \langle D\rangle C(x) \in S$. Then we obtain S' by adding $\gamma \mid D : C(x)$ to S, which must be satisfied in $\mathcal{C}$. Hence (1) holds. For condition (2), we set $\pi(\gamma \mid D) := w$ for $w \in W$ such that $w = D^{\mathcal{I}_{\pi(\gamma)}}$, and leave σ unmodified.
- Suppose we apply $\Rightarrow_\supset$ to some $\gamma \mid C : \phi(x) \in S$. Then we obtain S' by adding $\gamma : C(x)$ to S, which must be satisfied in $\mathcal{C}$. Hence (1) holds. For condition (2) we observe that the same mappings π and σ are sufficient.
- Suppose we apply $\Rightarrow_{\neq}$ to some $\gamma : C(x) \in S$ and $\delta : \neg C(x) \in S$. Then we obtain S' by adding $p(\gamma) \not\equiv p(\delta)$ to S, which, considering the character of translation p, must be satisfied in $\mathcal{C}$. Hence (1) holds. For condition (2) we observe that the same mappings π and σ are sufficient.

By Lemma 2 the number of inference steps applicable to S is finite, therefore at some point we obtain a complete constraint system, which is clearly clash-free. ❑